

Orbit dimensions in bosonic systems

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Bosonic quantum information platforms, such as quantum optics, constitute a major family of quantum systems currently being explored for building a universal quantum computer. These systems are associated to an infinite-dimensional Hilbert space, known as the *m-mode Fock space*. For quantum optics, these modes correspond to distinct properties (spectral, temporal, angle of polarization, etc.) of the electromagnetic field that are considered. This is the setting of both discrete variable (DV) [1, 2] and continuous variable (CV) [3–5] approaches. Ideally, such a quantum computing platform should be *universally controllable*, i.e. have the ability to unitarily evolve any state in Fock space into any other state, by a suitable driving sequence [6]. This is not the case for linear or even Gaussian quantum optics, as the group of accessible unitaries G is finite-dimensional and therefore *sub-universal*: it cannot make an initial quantum state explore the infinite-dimensional state space. In fact, since $\dim(G)$ is only *quadratic* in the number of modes, it cannot implement arbitrary unitary transformations deterministically on some subspace of Fock space of finite but *exponentially* large dimension, either [7]. Recovering universality thus requires the ability to enact additional non-Gaussian unitaries [3] (which is experimentally challenging) or to use intermediate measurements with post-selection [8] (which trades determinism for low success rates). Other strategies using measurements to increase the expressivity of gates include adaptive gates [9] (which increases delay and losses) or adaptive state injection [10] (which sacrifices purity). In the meantime, sub-universal bosonic platforms can still provide interesting quantum information processing uses, such as the celebrated Boson Sampling task [2, 5], but also in providing quantum machine learning (QML) ansätze that may escape classical simulation techniques while remaining trainable [11–14].

While most work looking at advantageous QML tasks has focused on finite-dimensional systems, there is strong recent interest in leveraging the CV bosonic systems, as the inherent infinite-dimensionality of the Hilbert space allows technically for a wider classification of dynamical unitary groups [15]. Besides, the most widespread metric of expressivity for qubit-based QML models, namely closeness to t -designs [12, 13, 16], cannot be naively ported to CV bosonic models (as definitions of t -designs or even Haar measures over unitaries can become too stringent there [17]); this reveals a need to seek for other meaningful measures of expressivity that would find a unified footing for both DV and CV-based QML models.

In QML applications, the choice of the initial state $|\psi\rangle$ may greatly impact the richness of the quantum model obtained, but in a way that is not always well understood. Incidentally, a platform with a sub-universal unitary group G can only send a prepared state $|\psi\rangle$ to a *strict* submanifold of state space, called the *orbit* of $|\psi\rangle$ under G . Furthermore, the preparation of specific bosonic states $|\psi\rangle$, and thus the experimental access to certain orbits, remains a delicate case by case issue [18]. Different preparable quantum states can hence behave as different resources, in regard to the extent of their accessible state space, i.e. their orbit under G . Therefore, to better understand both the accessible state space of the quantum platform, and the expressivity of a variational circuit built on it for a (QML, metrology, Boson sampling... etc.) application, it appears of crucial importance to gain understanding of the structure of the platform's orbits, and of how they partition the state space. In that regard, there exist in the literature several *invariants* under passive linear optics [19–22] and Gaussian optics [23, 24] that have been identified. While somewhat interpretable, and useful to rule out certain state conversions or to verify experimental setups [25], these invariants

do not always help to convey a better picture of the orbits themselves.

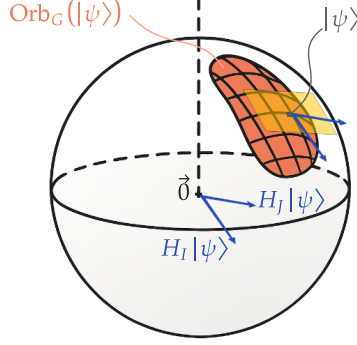


FIG. 1. Illustration of an orbit of a state $|\psi\rangle$ under group G (orange). The tangent space to the orbit at point $|\psi\rangle$ (yellow) encodes the possible movement directions (blue) under small evolutions from the group G . The vectors $H_I|\psi\rangle$ ($H_I \in \mathcal{B}_{\mathfrak{g}}$, a basis of the Lie algebra $\mathfrak{g}$ of the group G) are translations of a basis of these directions to the origin of the Hilbert space. The dimension of the orbit, equal to the tangent space dimension, is found by counting the number of linearly independent (w.r.t. *real* linear combinations) vectors $H_I|\psi\rangle$.

In this work, we propose to study the *manifold dimension* of a quantum state’s orbit (as a manifold in the Hilbert space), a simple yet nontrivial property quantifying “how many” states it can explore in state space.

By leveraging the Lie-algebraic structure of the underlying metaplectic group, we provide a general straightforward way to compute orbit dimensions (see Fig. 1) for states specified either in a DV language (coefficients in the Fock basis) or in a CV language (phase-space representations such as a Wigner or stellar functions). This can be tackled numerically (or with symbolic computation), but also analytically for highly structured states. Notably, our work treats both DV and CV settings at once under a common framework, and as such we rigorously deal with the mathematical technicalities that arise when *finite*-dimensional Lie groups act on the *infinite*-dimensional Fock space.

We also formally prove that these orbit-dimensions quantities are indeed upper-bounds for the number of directions explored by bosonic variational quantum circuits, as well as for the rank of the Quantum Fisher Information Matrix. This reinforces their use as a measure of expressivity, which for technical reasons was less immediate than in the finite-dimensional (qubit) case. When trying to design a quantum encoder of classical data made of p features via a choice of VQC ansatz, the orbit dimension thereby indicates the maximal value of p for which the encoder may (locally) work without loss of information.

To illustrate our framework, we obtain the orbit dimensions different families of states: Fock basis states, single-mode superposition states, and NOON states. Interestingly, we find that orbit dimensions of Fock basis states $|\mathbf{n}\rangle = |n_1, \dots, n_m\rangle$ do not depend on the particular values of the occupation numbers n_i , but only on whether they are zero or non-zero. This reveals that while boson can (unlike fermions) bunch, doing so does not increase the number of accessible directions in state space.

Besides revealing a direct topological property of the attainable state spaces, orbit dimensions can also be used to show the impossibility of some state conversions or of implementing certain logical gates on logical subspaces. We illustrate this with an impossibility result for deterministically implementing the infamous CNOT gate in the dual-rail encoding [8].

We prove that *generic* bosonic states (according to suitable measures on state space) have maximal orbit dimensions (except in edge cases), i.e. equal to $\dim(G)$, and provide constructive

examples of such families. We also prove a one-way robustness result, stating that when perturbing slightly a state (according to a physically suitable topology), its new orbit dimension will be at least as large as before.

Additionally, we discuss the possibility of efficiently evaluating orbit dimensions experimentally given many copies of an unknown state (without the need for full tomography). For this, we propose a first protocol for pure states which relies on single-copy access and a polynomial amount of homodyne measurements; and a second general protocol that relies on two-copy access and photon-number resolving detectors, using a modification of the bosonic SWAP-test. A spectral-gap type dependency does however appear in the associated sample complexities of these schemes, which hinders at efficiency of these measurement protocols, without further partial assumptions on the states.

Next, we prove a (to our knowledge novel) connection with the resource theory of non-Gaussianity [26–28]: *free* (i.e. Gaussian) pure states over $m \geq 2$ modes coincide exactly with the unique *orbit of minimal dimension* under m -mode Gaussian unitaries. We indeed establish in this setting the lower bound $\dim(\text{Orb}(|\psi\rangle\langle\psi|)) \geq m(m+3)$, with equality occurring for *if and only if* $|\psi\rangle\langle\psi|$ is Gaussian.

Lastly, we establish connections (in the form of inequalities) between these concepts of orbit dimensions, and the widely used notions of stellar rank [23, 24], symplectic eigenvalues [29] and symplectic rank [29, 30].

Overall, our work lays the groundwork for proper understanding of the dimensional expressivity of bosonic QML models, by showcasing a Lie algebraic framework to obtain the dimension of reachable orbits that remains rigorously backed in the infinite-dimensional Hilbert space setting of CV quantum optics. This measure of expressivity of the Gaussian orbit dimension, which is found to be minimal for Gaussian states and maximal for generic states, can also be regarded as a quantifier of the asymmetry resource in the state [31] (with Gaussian states being the most symmetrical, and Haar random subspace states being the most asymmetrical). Orbit dimensions can not only help the QML practitioner understand the sources and limitations of expressivity in their bosonic QML model, but our proposed schemes to probe them experimentally may provide them with a practical benchmarking tool to assess the quality of their initial state’s expressivity.

Note: A previous version of this work is available on arXiv at <https://arxiv.org/abs/2506.07995>.

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