

Variational (matrix) product states for combinatorial optimization

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Abstract

To compute approximate solutions for combinatorial optimization problems, we describe variational methods based on product-state (PS) and matrix-product-state (MPS) ansatzes. We perform variational energy minimization with respect to a quantum annealing Hamiltonian and utilize randomness by embedding the approaches in the metaheuristic iterated local search (ILS). The resulting quantum-inspired ILS algorithms are benchmarked on maximum cut problems of up to 50 000 variables. We show that they can outperform traditional PS/MPS methods, classical ILS, the quantum approximate optimization algorithm, and other variational quantum-inspired solvers.

1 Introduction

Combinatorial optimization problems arise across a wide range of scientific and industrial applications, including logistics, scheduling, communication networks, and resource allocation [1–4].

From the perspective of quantum computing, quantum annealing [5–9] and variational quantum algorithms [10–12] have emerged as promising approaches to combinatorial optimization. In particular, the Quantum Approximate Optimization Algorithm (QAOA) [13, 14] has become one of the most widely studied frameworks for solving optimization problems on quantum hardware. At the same time, current quantum devices remain limited by noise and scale, motivating the development of quantum-inspired algorithms that can exploit concepts from quantum computation while running efficiently on classical hardware.

Tensor networks provide a natural framework for such quantum-inspired approaches. Matrix-product states (MPS) [15–19] are powerful tensor-network variational representations that have achieved remarkable success in the classical simulation of quantum many-body systems [20, 21]. Their ability to efficiently represent and optimize high-dimensional states raises the question of whether tensor-network methods can also serve as effective optimization algorithms for large-scale combinatorial problems.

In this work, we investigate tensor-network-based variational optimization methods and benchmark them on the MaxCut problem [22, 23]. We perform energy minimization with respect to a quantum annealing Hamiltonian and combine the resulting optimization procedure with the classical metaheuristic iterated local search (ILS) [24]. This leads to the quantum-inspired iterative local search (QiILS) algorithm. We additionally introduce a parallel variant, quantum-inspired iterative global search (QiIGS), which employs global gradient updates and is naturally suited to GPU acceleration.

We benchmark the proposed methods on standard MaxCut benchmarks and random regular graph ensembles. We show that QiILS outperforms classical ILS, local quantum annealing

(LQA) [25], and generalized group-theoretic coherent states (GCS) [26–28] approaches on the considered benchmarks, while also achieving higher approximation ratios than the QAOA benchmarks considered in this study. Furthermore, QiIGS enables efficient large-scale optimization through parallel execution. These results highlight the potential of tensor-network-based variational methods as scalable quantum-inspired optimization algorithms.

2 Quantum-Inspired Tensor-Network Optimization

We consider the MaxCut problem, which can be mapped onto the Ising Hamiltonian [29, 30]

$$H_f = \sum_{\langle j,k \rangle} w_{j,k} \sigma_j^Z \sigma_k^Z. \quad (1)$$

Following the quantum annealing paradigm [5–9], we introduce the interpolation

$$H(\lambda) = (1 - \lambda)H_i + \lambda H_f, \quad H_i = - \sum_{j=1}^n \sigma_j^X. \quad (2)$$

Low-energy states of $H(\lambda)$ are approximated using variational product-state (PS) and MPS ansatzes. The PS corresponds to bond dimension $\chi = 1$, while larger bond dimensions allow the MPS to represent entangled states at increased computational cost.

The first algorithm, QiILS, combines variational tensor-network optimization with iterated local search. Starting from a randomly initialized (M)PS, we minimize the energy at fixed λ , extract a candidate bitstring, evaluate its cut value, and retain the best solution found so far. The optimized state is then perturbed by randomly applying spin flips to a fraction p of its sites, and the procedure is repeated. Thus, the quantum-annealing Hamiltonian shapes the local optimization landscape, while the perturbation step explores different regions of the solution space.

For the PS case, the variational state can be written as

$$|\Psi\rangle = \bigotimes_{j=1}^n [\cos(\theta_j)|0\rangle + \sin(\theta_j)|1\rangle], \quad \theta_j \in \left[0, \frac{\pi}{2}\right]. \quad (3)$$

This representation leads to efficient angle-update rules and makes the method particularly attractive for large problem instances. Motivated by this observation, we also introduce QiIGS, which replaces the sequential local updates of QiILS by parallel gradient updates of the form

$$\theta_j \leftarrow \theta_j - \tau \frac{\partial E}{\partial \theta_j}, \quad (4)$$

where $E = \langle \Psi | H(\lambda) | \Psi \rangle$ and τ is a step size. Since all angles can be updated simultaneously, QiIGS is naturally suited to GPU acceleration and provides a scalable variant of the tensor-network-inspired optimization framework.

3 Benchmark Results

We benchmark the proposed methods on standard MaxCut benchmarks and random regular graph ensembles. Performance is measured through the approximation ratio r .

Figure 1(a) compares QiILS with QAOA results reported in Ref. [14]. For QAOA, performance is quantified by the number of circuit layers q , while for QiILS we consider either the number of

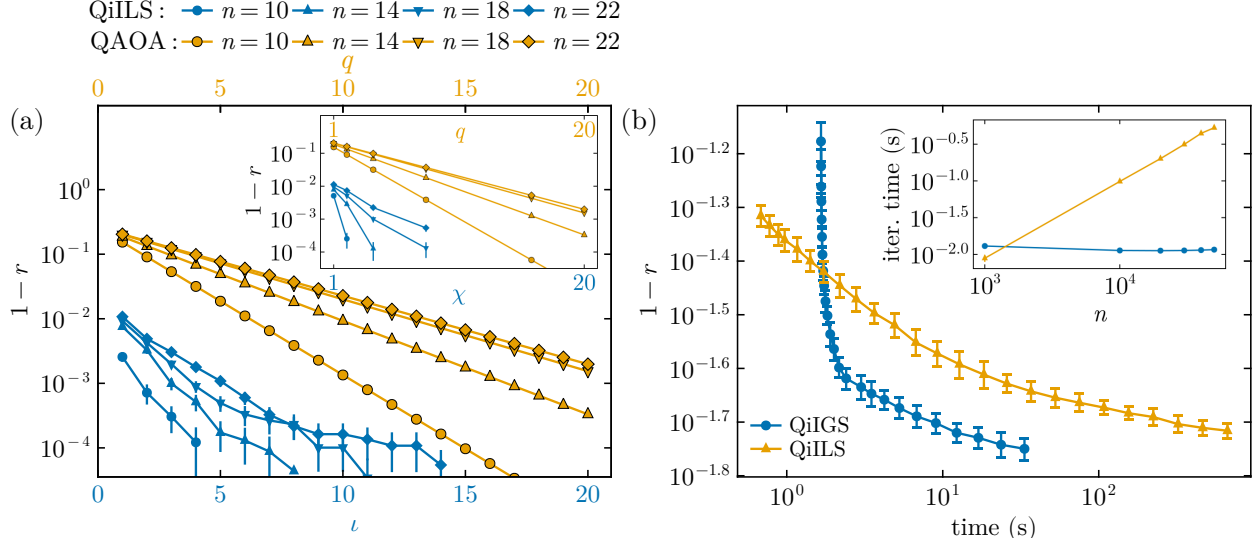


Figure 1: Benchmark results for the proposed tensor-network optimization methods. (a) Comparison of QiILS with QAOA benchmarks reported in Ref. [14]. Performance is measured by the average relative error $1-r$ over 1000 unweighted 3-regular random graph instances. The main panel shows QiILS as a function of the iteration number ι at fixed bond dimension $\chi = 1$, while the inset shows the dependence on the bond dimension χ for a single iteration. (b) Comparison of QiIGS and QiILS on the G81 benchmark instance. The main panel shows the average relative error $1-r$ as a function of wall-clock time, averaged over 20 random initial states. The inset reports the average iteration time on random 3-regular graphs with up to 50 000 variables, illustrating the scalability of QiIGS through parallel parameter updates.

iterations ι at fixed bond dimension $\chi = 1$, or the bond dimension χ for a single iteration. Although not directly equivalent, these quantities provide analogous measures of computational effort.

For unweighted 3-regular graphs, QiILS achieves higher approximation ratios than the QAOA benchmarks reported in Ref. [14], both as a function of iteration number and bond dimension. Increasing either the number of iterations or the bond dimension systematically improves performance, demonstrating the effectiveness of the variational tensor-network optimization approach.

To investigate scalability, we compare QiIGS and QiILS in Fig. 1(b). On the G81 benchmark instance [31], QiIGS achieves solution qualities comparable to those of QiILS while benefiting from parallel parameter updates. The inset shows the average iteration time for problem sizes up to 50 000 variables. These results demonstrate that the proposed tensor-network-inspired optimization framework can be applied efficiently to large-scale combinatorial optimization problems.

4 Discussion and Conclusion

We have presented QiILS and QiIGS, two quantum-inspired optimization algorithms based on variational (M)PS representations. By combining tensor-network variational optimization with stochastic search, QiILS achieves higher approximation ratios than the QAOA benchmarks considered in this study and outperforms several existing quantum-inspired and classical optimization approaches. The parallel-update structure of QiIGS further enables efficient optimization on instances containing up to 50 000 variables. These results highlight tensor-network-based variational methods as scalable quantum-inspired algorithms for combinatorial optimization.

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