

**Extended abstract for QTML 2026 submission:
Witness expansion: a unified framework for analytical and measurable mixed-state
resource detection**

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BACKGROUND AND MOTIVATION

A key aspect of quantum information science is to characterize “quantumness”, especially through identifying which objects can be recognized as “resources” for information-processing tasks [1]. The relevant quantum resource is defined for various tasks and on different physical models. For example, coherence supports phase-sensitive protocols and quantum random number generation [2, 3]; entanglement is central to many-body physics, secure communication, and simulation [4, 5]; non-stabilizerness, also known as magic, marks the extra ingredient that upgrades Clifford computation to universal quantum computation [6–8]; and fermionic non-Gaussianity plays the analogous role beyond matchgate or free-fermionic dynamics [9, 10]. A useful quantum device should therefore be able to create, preserve, and manipulate the appropriate resource in noisy mixed-state regimes. Detecting resources is consequently a basic step in benchmarking devices, diagnosing many-body dynamics, and locating the boundary between efficient classical simulation and quantum advantage.

Resource detection is both operational and conceptual: it certifies the ingredient needed for a task and clarifies which structural feature of a state contributes to the “quantumness”. One obstacle is that free sets are usually defined by convex hulls or orbit closures rather than by a simple equation: a state is separable if it is a mixture of product states; stabilizer if it is a mixture of stabilizer pure states; and convex-Gaussian if it is a mixture of pure fermionic Gaussian states. These definitions are conceptually clean but hard to test directly, and exact membership or detection can be computationally hard even when the density matrix is known [11, 12]. The theory of entanglement detection already provides methods that convert a hard geometric question into measurable signatures: witnesses, PPT, entropy, and moment criteria [13–18]. However, extending the success of entanglement detection criteria to other quantum resources is far from straightforward. Many powerful tools for entanglement detection are tailored to the geometry of separability and do not directly provide a general methodology for detecting resources with different free-state structures. Thus, to build a universal quantum resource detection framework beyond pure states [19], one should identify features shared by general resource theories rather than rely on resource-specific coincidences, and without rebuilding the theory from scratch for every resource.

FRAMEWORK

Our proposed solution, *Witness expansion* (WE), is a resource detection framework with the following properties: It is *universal*: the same construction applies to pure and mixed states and to different resource theories whenever a free-unitary symmetry is available. It is *analytical*: the resulting criteria are related to analytical quantities that can be derived in concrete theories. It is *measurable*: the involved quantities can be estimated by multi-copy randomized measurement protocols. It is *adjustable*: the parameters within tune the tradeoff between detection power, symmetry, and experimental cost. It is *faithful* for pure states: a single nonlinear criterion can detect all pure resource states.

The mechanism is illustrated in Fig. 1a. A linear witness W defines a hyperplane on which all free states lie on one side, and each free unitary U produces another valid witness $U^\dagger W U$. Instead of choosing a single rotated hyperplane, WE aggregates the whole orbit of witness expectations into an L^α -norm and compares it with the maximal value attainable by free states. It starts from familiar resource-theoretic input—a witness and a group of free unitaries—into a nonlinear, tunable, and experimentally accessible resource detection criterion.

For a linear witness W , its standardized witness $\tilde{W} = \mathbb{I} - W/\|W\|_\infty$, a group of free unitaries $\mathcal{G}$, and $\alpha \geq 1$, we define the WE α -norm of a state and the corresponding WE α -criterion as

$$\|\rho\|_{W,\mathcal{G},\alpha} = \left(\mathbb{E}_{U \in \mathcal{G}} \text{Tr}[U^\dagger \tilde{W} U \rho]^\alpha \right)^{1/\alpha}, \quad \|\rho\|_{W,\mathcal{G},\alpha} > C_{W,\mathcal{G},\alpha} \Rightarrow \rho \notin \mathcal{F}, \quad (1)$$

where the threshold $C_{W,\mathcal{G},\alpha}$ is the largest value attainable by free states in $\mathcal{F}$. The concrete results below show that this abstract recipe simultaneously recovers known resource diagnostics and produces new mixed-state criteria. Across all applications, we adopt a unified mechanism to derive nonlinear resource detection criteria: choose a witness, expand it with the free unitaries, compute the group twirl, and compare

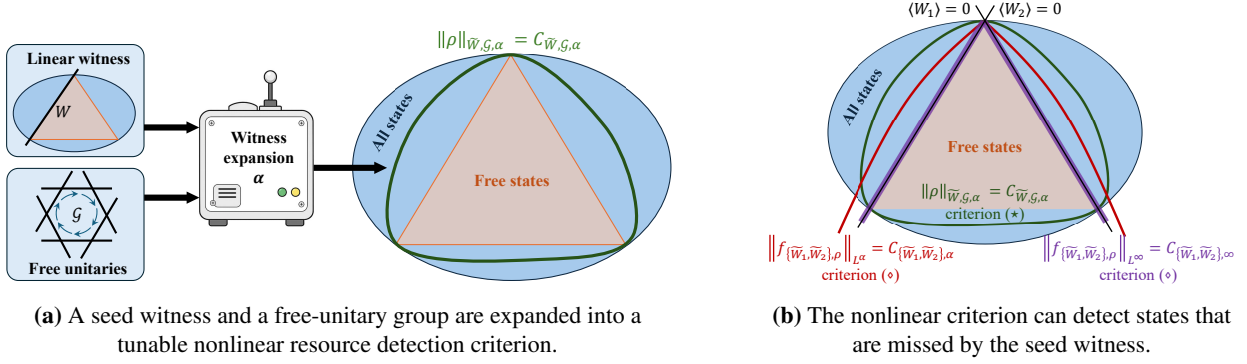


FIG. 1: Witness expansion turns a linear witness W and free unitaries $\mathcal{G}$ into a nonlinear criterion with tunable parameter α .

the resulting quantity with the free-state threshold. Figure 1b summarizes the idea: a seed hyperplane is expanded along free-unitary symmetries into a nonlinear boundary, and the criterion approaches the union of the orbit witnesses in the large- α limit.

The construction is valid for any convex resource theory equipped with a compact group of free unitaries. For an integer α , $\|\rho\|_{\bar{W}, \mathcal{G}, \alpha}^\alpha$ is the expectation value of an α -copy twirled operator, so it is both analytically tractable whenever the corresponding twirl is known and experimentally accessible as a multi-copy observable. The optimized threshold is often simple: if the pure free states form one orbit under $\mathcal{G}$, convexity and symmetry reduce the threshold calculation to a representative pure free state (**Observation 1** in the technical manuscript, same below). Moreover, the size of a violation provides a lower bound of the free robustness whenever $C_{\bar{W}, \mathcal{G}, \alpha}$ is nonzero (**Proposition 1**). Thus, a WE criterion gives both a yes/no certificate and a quantitative separation from free states.

The resource-specific results are shown below. Coherence calibrates the construction in a setting where the answer is already known. Entanglement shows that local-unitary expansion unifies entropy, moment, and PPT-like signatures. Magic demonstrates the role of design order and the tradeoff between full Clifford group and subgroup sensitivity, and provides a criterion based on high-magic states that detects all pure magic states. Fermionic non-Gaussianity shows how matchgate twirling converts a scalable fidelity witness into mixed-state tests for the convex hull of pure Gaussian states, and how an even-degree Majorana product leads to a criterion that is faithful for all pure states.

RESOURCE-SPECIFIC RESULTS

- **Coherence: complete detection with a two-copy criterion (Theorem 1).** For coherence, the free states are diagonal and the free unitaries are diagonal phase rotations. Expanding a single off-diagonal witness gives a closed formula for all α , with free-state threshold equal to one. At $\alpha = 2$ the criterion reduces to $\sum_{j \neq k} |\rho_{jk}|^2 > 0$, hence detects exactly all coherent states and recovers the familiar l_2 coherence norm [2]. This example shows that WE reproduces the standard resource quantifier when free geometry is simple.
- **Entanglement: entropies, moments, and a PPT-like limit (Theorems 2–3 and Corollary 1).** For bipartite systems $\mathbb{C}^d \otimes \mathbb{C}^d$, the seed witness is chosen as the SWAP operator and $\mathcal{G}$ is the local-unitary group. **Theorem 2** gives the all- α formula by Schur–Weyl duality and unitary Weingarten calculus, with a simple form of free threshold C_α . **Corollary 1** makes the first nontrivial cases explicit. The two-copy criterion is an inequality related to purities, which detects every pure entangled state and reduces on pure states to the second Rényi entanglement entropy. The three-copy criterion contains the partial-transpose moment $\text{Tr}[(\rho^{T_B})^3]$, the third moments of the marginals, and a correlation term $\text{Tr}[\rho(\rho_A \otimes \rho_B)]$, giving a systematic moment-based mixed-state test. **Theorem 3** identifies the $\alpha \rightarrow \infty$ limit with a PPT-like condition: a state is detected whenever $\langle \Phi | \rho^{T_B} | \Phi \rangle < 0$ for some maximally entangled $|\Phi\rangle$ in the local-unitary orbit.
- **Odd-prime-dimensional qudit magic: full Clifford and subgroup tests (Theorems 4–5).** For odd-prime local dimension, Wigner phase-space point operators provide canonical magic witnesses because stabilizer states have nonnegative discrete Wigner functions. Since the qudit Clifford group is a unitary 2-design, the first two full-Clifford scores depend only on ordinary state moments and cannot detect magic (**Observation 2**). The first nontrivial full-Clifford test appears at three copies, as stated in **Theorem 4**, which detects all pure non-stabilizer states. **Theorem 5** shows that a smaller translation subgroup already gives a nontrivial two-copy criterion based on one column of the Wigner function. The two criteria are genuinely complementary: the subgroup test detects some mixed states missed by the full three-copy test,

while the full Clifford test detects all pure magic states, including those not detectable by the two-copy criterion.

- **Qubit magic: fourth-order Clifford invariants, relation to stabilizer entropy, and a pure-state faithful criterion (Theorem 6 and Corollaries 2–3).** For qubits, the Clifford group is a unitary 3-design, so all full-Clifford criteria with $\alpha \leq 3$ are spectral and therefore trivial for magic detection (**Observation 2**). We derive a criterion using the fourth-order Clifford twirl for a general seed operator in **Theorem 6**, which is a Clifford-invariant combination of state moments and Pauli-moment quantities. As specific examples, we first use a witness based on the triangle criterion [20] as the seed and show its pure-state reduction as in **Corollary 2**. It relates directly to the Pauli fourth moment $\mathfrak{P}_2(\psi) = d^{-1} \sum_{P \in \mathcal{P}_n} |\langle \psi | P | \psi \rangle|^4$, equivalently to stabilizer 2-Rényi entropy [21], but the criterion becomes trivial for $n \geq 3$ qubits. Beyond this constraint on system size, in **Corollary 3** we use the projector onto a pure high-magic state Φ as the seed operator. Whenever the seed obeys $\mathfrak{P}_2(\Phi) < 4/(2^n + 3)$, the resulting four-copy criterion detects all pure qubit magic states for every system size and remains a valid mixed-state witness. The criterion depends on the seed only through the scalar $\mathfrak{P}_2(\Phi)$, and admissible seeds we construct explicitly in the appendix exist with gap $4/(2^n + 3) - \mathfrak{P}_2(\Phi) = \Theta(2^{-2n})$.
- **Fermionic non-Gaussianity: a scalable mixed-state test and a pure-state faithful criterion (Theorems 7–8 and Corollaries 4–5).** The fermionic application targets $\text{Conv}(\mathcal{G}_n)$, the convex hull of pure fermionic Gaussian states, which is the free set naturally associated with matchgate and fermionic linear-optics simulation. In **Theorem 7**, we use projectors onto an explicit family of n -qubit pure non-Gaussian states P_n to build a WE criterion. Every convex Gaussian state has an overlap of at most $2^{-\lfloor n/4 \rfloor}$ with these projectors, so they are scalable, resource-sensitive seeds. Matchgate twirling provides explicit WE 2- and 3-norms in terms of Majorana sector purities $B_\ell(\rho)$ and three-sector matchgate invariants, with exact convex-Gaussian thresholds. On pure physical states with $4 \leq n \leq 7$, the WE 2-norm is a function of the first-order fermionic antifatness [22] and is faithful for pure non-Gaussianity (**Corollary 4**). In **Theorem 8**, we use an even Majorana product of degree 2ℓ as a Hermitian seed to obtain a WE 2-criterion that remains nontrivial for any system size. Especially, the criterion with the central sector $\ell_* = \lfloor n/2 \rfloor$ is faithful on every fixed-parity pure state: $B_{\lfloor n/2 \rfloor}(\psi) \geq \binom{n}{\lfloor n/2 \rfloor}$, with equality exactly for pure Gaussian states (**Corollary 5**). To our knowledge, this is the first analytical arbitrary-size mixed-state criterion for detecting states outside $\text{Conv}(\mathcal{G}_n)$ that is still faithful on all pure physical fermionic states.

Cross-resource consequences. The above results identify two observations. First, the minimum number of state copies for a nontrivial criterion is controlled by symmetry: coherence and entanglement already have informative two-copy tests, whereas magic requires going beyond the Clifford design order or reducing the symmetry. Second, some well-known pure-state resource measures arise as reductions of mixed-state certificates: Rényi entanglement entropy, stabilizer Rényi entropy, and fermionic antifatness all appear in this way. Third, by choosing the seed operator properly, the resulting WE criteria are faithful for all pure states in all the above cases we discuss, while they are still capable of detecting mixed resource states.

CONCLUDING REMARKS

Witness expansion provides a reusable route from linear witnesses to nonlinear resource certificates. From the same small amount of input—a seed witness and a group of free unitaries—it recovers coherence norms, entanglement Rényi quantities, partial-transpose moments, stabilizer Rényi entropy, and fermionic antifatness, and also gives new criteria for mixed-state qudit magic, qubit magic, and fermionic non-Gaussianity.

The main impact is that several resource theories can be treated in one operational template. Entanglement, magic, and fermionic non-Gaussianity describe different obstructions to classical simulation. The quantity related to the resource detection criterion, i.e., the WE norm, is a polynomial function of a density matrix, whose algebraic form is determined by a twirl, directly connecting resource detection with commutants and unitary design theory. These areas are central to randomized benchmarking, classical shadows, magic theory, and matchgate simulation.

The practical design rule is also clear. Large unitary groups give clean formulas and strong pure-state statements, while smaller subgroups can preserve witness directions that are more sensitive to noisy mixed states. Higher copy number can overcome obstructions from the symmetry of the group of free unitaries, as we see in detecting quantum magic, but may increase experimental cost. The resulting hierarchy is therefore a menu of interpretable certificates matched to the resource, the platform, and the available measurement budget.

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