

Triangle Criterion: a mixed-state magic criterion with applications in distillation and detection

I. INTRODUCTION

Entanglement and magic (or non-stabilizerness) are central non-classical resources that empower quantum communication and computation beyond what is possible classically [1, 2]. Entanglement has been studied for decades, leading to a well-developed toolbox of entanglement measures and witnesses that clarify both the geometry of entangled states and their operational role in information-processing tasks [3, 4]. By contrast, the theory of magic is comparatively young: after the Gottesman–Knill theorem [5], it became clear that non-stabilizer “magic” states are the computational resources that turn classically simulable stabilizer circuits into universal quantum computers [6–8]. Despite the surging recent interest and efforts on magic theory, most progress has focused on the quantification aspect within the framework of resource theories [8–12]. Simple and physically meaningful tools for characterizing magic, especially for general mixed states, are still largely lacking.

A promising route is to draw inspiration from the mature tools developed for entanglement detection, among which the Positive Partial Transposition (PPT) criterion [13] has emerged as a prominent one. In its simplest form, it asks whether the partial transpose ρ^{T_A} of a bipartite state ρ is positive semidefinite, where T_A denotes the partial transposition on subsystem A . Despite this concise mathematical formulation, the PPT criterion enjoys several key properties: (1) it provides a necessary and sufficient condition for separability of mixed states in 2×2 and 2×3 dimensions and for pure states in arbitrary dimensions [14]; (2) it is fundamentally linked to bound entanglement: any PPT state is non-distillable and hence cannot be used to prepare Bell states [15, 16]; (3) it yields a tight purity threshold for entanglement, $\text{Tr}(\rho^2) > 1/(d-1)$ with d being the dimension of ρ , so that all states below this threshold are guaranteed to be separable [17]; and (4) it is efficiently computable and can be estimated experimentally via multi-copy and single-copy protocols [18–22]. Because of these favourable properties, the PPT criterion has been widely adopted in diverse tasks, including the resource theory of entanglement [23, 24], quantum device benchmarking [25], and quantum field theory [26]. It is thus natural and important to ask: *is it possible to construct a single, mixed-state magic criterion that reproduces the key advantages of the PPT criterion?*

In this work, inspired by various previous results on magic distillation and stabilizer polytopes [27–29], we establish the *Triangle Criterion*, a mixed-state magic criterion taking the form of fidelity triangle inequalities, with following properties closely mirroring those of the PPT entanglement criterion: (1) it provides a necessary and

sufficient condition for detecting magic in single-qubit mixed states and in pure states of an arbitrary number of qubits; (2) a multi-qubit mixed state can be converted into a single-qubit magic state by stabilizer operations (including post-selection) if and only if it is detected by the criterion; (3) it yields an upper bound on the minimal purity of magic states, $1/(d-1/2)$, which is conjectured to be tight with numerical and constructive evidences; and (4) it can be estimated experimentally. Despite these similarities, a key difference from PPT is that the Triangle Criterion is not tensor-stable: the tensor product of two states that each satisfy the Triangle Criterion can violate it when viewed as a larger system, leading to non-trivial consequences for magic distillation.

With this magic criterion, we identify a family of two-qubit states that are provably useless for any single-qubit magic distillation protocol: under arbitrary stabilizer processing, every single-qubit state obtainable from one copy is a stabilizer state. Nevertheless, we construct a distillation routine that converts multiple copies of these two-qubit states into T states, providing (to our knowledge) the first proof that genuinely multi-qubit magic distillation protocols are strictly more powerful than all single-qubit schemes. We further show that, for random mixed states with rank polynomial with qubit number, single-qubit magic distillation protocols fail with high probability. This indicates that single-qubit distillability of multi-qubit states is fragile to noise. We further establish the existence of *unfaithful magic states*, which cannot be detected by any fidelity-based magic witness. In addition, our conjectured lower bound on the minimal purity of magic states implies both the existence of *absolute stabilizer states*, states that cannot be transformed into magic states by any unitary evolution and measurement post-selection, and a *fundamental limitation* on certifying the magic of mixed states.

We will frequently use the following mixed-state distribution [30]:

Definition 1. A d -dimensional mixed state ρ is sampled from the distribution $\pi_{d,k}$ when $\rho = \text{Tr}_k(|\Psi\rangle\langle\Psi|)$ where $|\Psi\rangle$ a $d \times k$ -dimensional pure state sampled from Haar measure.

II. TRIANGLE CRITERION

Inspired of the geometry of single-qubit stabilizer polytope [31], we formulate the following general magic criterion:

Theorem 1 (Triangle Criterion). *Given any quantum state ρ , if there exist three stabilizer pure states*

$\{\psi_1, \psi_2, \psi_3\}$ with $\text{Tr}(\psi_i \psi_j) = 1/2$ for all $i \neq j$ such that

$$\text{Tr}(\rho \psi_1) > \text{Tr}(\rho \psi_2) + \text{Tr}(\rho \psi_3), \quad (1)$$

then ρ is not a mixture of stabilizer states. This criterion detects all single-qubit mixed magic states and all multi-qubit pure magic states.

In the single-qubit case, the validity of this criterion is geometrically clear: the equalities $\text{Tr}(\rho \psi_1) = \text{Tr}(\rho \psi_2) + \text{Tr}(\rho \psi_3)$ describe all facets of the stabilizer octahedron with vertices $|\pm\rangle \sim |0\rangle \pm |1\rangle$, $|0/1\rangle$, and $|\pm i\rangle \sim |0\rangle \pm i|1\rangle$. For the general multi-qubit case, the proof follows from its connection to magic distillation, as explained in the next section.

Based on the fidelity with stabilizer pure states, the Triangle Criterion can potentially be efficiently verified with techniques of Refs. [32, 33]. Besides, we prove that the Triangle Criterion is equivalent with testing a number of $4(2^n - 1)2^n \prod_{l=1}^n (2^l + 1) \leq 2^{\mathcal{O}(n^2)}$ low-rank magic witnesses

$$W_{ijk} = \psi_i + \psi_j - \psi_k. \quad (2)$$

So the classical shadow protocol with global random Clifford unitaries [34] can simultaneously estimate all these observables to constant error rates with sample complexity of order $\mathcal{O}(n^2)$ with only single-copy operations.

III. MULTI-QUBIT MAGIC DISTILLATION

The most crucial property of the Triangle Criterion is its connection with magic distillation, which starts from the following result:

Theorem 2. *Given an n -qubit state ρ , it can be transformed into a single-qubit magic state with Clifford unitary rotations, Pauli measurements, and stabilizer ancillas if and only if it can be detected by Triangle Criterion.*

This theorem can be proved using two key ingredients. First, we show that the Triangle Criterion is equivalent to estimating the magic witness

$$W_U = U [(|+\rangle\langle+| + |+i\rangle\langle+i| - |0\rangle\langle 0|) \otimes |0^{n-1}\rangle\langle 0^{n-1}|] U^\dagger, \quad (3)$$

where U is chosen from the entire Clifford group. Notice that $|+\rangle\langle+| + |+i\rangle\langle+i| - |0\rangle\langle 0|$ is itself a magic witness and, after suitable single-qubit Clifford rotations, can be used to detect all single-qubit magic states. Estimating $\text{Tr}(W_U \rho)$ is therefore operationally equivalent to first rotating ρ with a Clifford unitary, then projecting $n-1$ qubits onto the state $|0^{n-1}\rangle$, and finally testing the magic of the resulting single-qubit state. Consequently, if a state is detected by the Triangle Criterion, there exists a stabilizer operation that reduces it to a single-qubit magic state. Since Clifford unitaries and projections onto $|0^{n-1}\rangle$ cannot create magic, this also establishes the correctness of the Triangle Criterion in the multi-qubit setting.

The second ingredient comes from Lemma 1 of Ref. [28], which shows that if a stabilizer procedure (with measurement post-selection and stabilizer ancillas) can map an n -qubit state ρ into a single-qubit magic state, then there exists another procedure that maps ρ to a single-qubit magic state and consists solely of an n -qubit Clifford unitary followed by projecting $n-1$ qubits onto $|0^{n-1}\rangle$, which is equivalent to estimating Eq. (3). We can therefore conclude that, if there exists a stabilizer operation that maps ρ to a single-qubit magic state, then ρ can be detected by the Triangle Criterion, which completes the proof of Theorem 2. Moreover, Theorem 2 also implies that the Triangle Criterion detects all pure magic states, since any multi-qubit pure magic state can be reduced to a single-qubit pure magic state by a suitable stabilizer procedure [27].

Note that Theorem 2 is similar with the relationship between PPT and entanglement distillation: a PPT entangled state cannot be used for preparing Bell states. A major difference between PPT and Triangle criteria is the stability under tensor product. It is easy to show that if two states are both PPT, then the tensor product of these two states is also PPT. This means that there exists some bound entangled state which cannot be transformed into Bell states given arbitrary number of copies. However, this conclusion cannot be directly generalized to Triangle Criterion as the definition of magic has no space structure, i.e., it is not compatible with the tensor product operation. Actually, we prove that the Triangle Criterion is not tensor stable:

Theorem 3. *There exists a two-qubit state which cannot be converted into single-qubit magic states with stabilizer operations using a single copy. However, the state can be used to distill single-qubit T states given multiple copies.*

This conclusion indicates that genuinely multi-qubit magic distillation protocols are strictly more powerful than single-qubit distillation protocols, thereby resolving the open problem posed in Ref. [28]. We establish this result by numerically searching over all two-qubit states that are not detected by the two-qubit Triangle Criterion but are detected by the four-qubit Triangle Criterion with two copies. By optimizing the violation of the four-qubit Triangle Criterion, we find a two-qubit state such that two copies of this state can be converted into a single-qubit magic state, which can then be further distilled into a T state [6].

Having seen that single-qubit magic distillation protocols can fail for some multi-qubit states, a natural question is how often single-qubit protocols are actually useful. For a given multi-qubit state, a single-qubit distillation protocol can be useful only if the state can be reduced to a single-qubit magic state by stabilizer operations (the converse would rely on resolving the open problem of the existence of bound magic under arbitrary numbers of copies [35]). Since Theorem 2 shows that the Triangle Criterion is a necessary and sufficient condition for a multi-qubit state to be reducible to a single-qubit

magic state, we can sample states according to the distribution $\pi_{d,k}$ and compute the probability that they are detected by the Triangle Criterion. Leveraging tools developed in Ref. [36], we can prove:

Theorem 4. *Given a $d = 2^n$ -dimensional state ρ sampled according to the distribution of $\pi_{d,k}$, the probability that it can be distilled into single-qubit T state by some single-qubit magic distillation protocol is upper bounded by $\exp\{c_1 n^2 - c_2 k\}$, where c_1 and c_2 are constants.*

Since k upper-bounds the rank of ρ , it implies that when the rank of a random mixed state reaches order $\mathcal{O}(n^2)$, single-qubit distillation fails with overwhelming probability. In stark contrast, magic can persist even for k on the order of 2^n [37], exhibiting an exponential gap between the existence of magic and its distillability by single-qubit protocols.

IV. MINIMAL PURITY

In entanglement theory, an important conclusion is that the minimal purity of entangled d -dimensional state is arbitrarily close to $\frac{1}{d-1}$, which plays vital roles in entanglement detection [38] and geometric analysis [39]. The upper bound is given by the PPT criterion while it has also been shown to be the lower bound in Ref. [17]. Despite the distinct structures associated with entanglement and magic theories, we use the Triangle Criterion to establish a remarkably similar characteristic:

Theorem 5. *There exists a multi-qubit magic state with purity arbitrarily close to $\frac{1}{d-1/2}$.*

This theorem is proved by exhibiting a state on the surface of the polytope defined by the Triangle Criterion, i.e., with $\text{Tr}(W_{ijk}\rho) = 0$, that has minimal purity. A representative state is

$$\rho = \frac{1}{d-1/2} I^{\otimes n} + \frac{1/2}{d-1/2} (|0\rangle\langle 0| + |+\rangle\langle +| + |i\rangle\langle i| - 2I) \otimes |0^{n-1}\rangle\langle 0^{n-1}|. \quad (4)$$

We prove that this state actually lies on the boundary of the stabilizer polytope, in agreement with the conclusion of Ref. [29]. Hence, if we move this state towards the maximally mixed state in order to further reduce its purity, it will eventually enter the stabilizer polytope and become a stabilizer state. At the same time, one can check that this density matrix is strictly positive. Therefore, it is not on the boundary of the state space, and there exists a magic state outside the stabilizer polytope that is arbitrarily close to it. We conjecture that:

Conjecture 1. *Any d -dimensional multi-qubit state ρ with purity less equal than $\frac{1}{d-1/2}$ can be written as a mixture of stabilizer pure states.*

In single and two-qubit cases, we can numerically enumerate all different surfaces of stabilizer polytopes to test the correctness of our conjecture [28, 40]. For even more qubits, we have numerical evidence for the conjecture up to $n = 4$. It is worth mentioning that, although $\frac{1}{d-1/2}$ is close to $\frac{1}{d}$, the constant $1/2$ is crucial: it leads to the fundamental limitation in magic detection shown in Theorem 7, whereas this limitation would no longer hold if the constant were exponentially small in n .

V. MULTI-QUBIT MAGIC DETECTION

Witnesses are a powerful and experimentally attractive tool for resource detection, corresponding to hyperplanes that separate free and resourceful states. A common class of witnesses is based on fidelity, $W = \alpha I - |\psi\rangle\langle\psi|$, built on the simple idea that a resourceful state should have high fidelity with some resourceful pure state [41]. For entanglement, $|\psi\rangle$ is typically chosen to be a maximally entangled state and α is determined by the Schmidt coefficients of $|\psi\rangle$. For magic, $|\psi\rangle$ can be taken as a highly magical pure state and α as its maximal fidelity with a stabilizer pure state. In entanglement detection, it has been shown that there exist unfaithful entangled states that cannot be detected by any fidelity-type witness [42, 43], revealing that some entangled states are far from all entangled pure states. We find such a phenomenon also for magic and construct an unfaithful magic state using Theorem 5.

Theorem 6. *There exist magic states that cannot be detected by any magic witness of the form $\alpha I - |\psi\rangle\langle\psi|$.*

Actually, magic detection is not only hard using fidelity-type witnesses. If Conjecture 1 holds, then magic detection is generally subject to fundamental limitations, which would hold for all linear protocols even beyond magic witnesses. Here, by a linear protocol we mean that one decides whether the target state ρ is a magic state or not from the expectation values of M different observables, $\{\text{Tr}(O_i\rho)\}_{i=1}^M$. We prove that for mixed states sampled with $\pi_{d,k}$, a number of $\tilde{\Omega}(k)$ linear observables is required to detect magic with constant probability [36]. Note that k is the dimension of the traced-out system, which can be exponentially large in practical scenarios.

Theorem 7. *If Conjecture 1 is correct, then there exists a constant $c_1 > 0$ such that for any magic witness, the probability of successfully detecting magic for a mixed state sampled from $\pi_{d,k}$ is upper bounded by $\exp\{-c_1 k\}$. Moreover, there exists a constant $c_2 > 0$ such that the probability that the magic of the state can be detected from the expectation values of M different observables is upper bounded by $\exp\left\{M \ln\left(4\sqrt{Md}\right) - c_2 k\right\}$.*

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