

QAOA-MaxCut has barren plateaus for almost all graphs

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Variational quantum algorithms (VQAs) are based on a hybrid quantum-classical framework, where an initial quantum state ρ is processed by a parameterized quantum circuit (PQC), and an observable O is measured on the output. The parameters θ are iteratively tuned via a classical optimizer, with the aim of minimizing a loss function $\ell(\rho, O; \theta)$.

The trainability of such algorithms can be affected by the presence of so-called barren plateaus (BPs) [MBS⁺18], a phenomenon in which the variance of the loss function (or its gradient) decays exponentially with the system size. Thus, the optimization landscape becomes flat and featureless, which can render the training of VQAs intractable.

Barren plateaus can have many causes, including the expressiveness of the PQC, the randomness of the initial state, the locality of measurements, and hardware noise. These disparate causes were recently shown [RBS⁺24, FHC⁺24] to share the same underlying mathematical explanation in terms of the Dynamical Lie algebra (DLA), a vector space generated by the set of operators corresponding to the parameterized gates in the circuit. For an n -qubit PQC, the DLA $\mathfrak{g}$ is a subalgebra of the Lie algebra $\mathfrak{u}(2^n)$, and decomposes into a direct sum of an abelian center and a number of simple Lie algebras $\mathfrak{g}_j$. The variance of the loss function can be precisely characterized by the projections of ρ and O onto the simple components $\mathfrak{g}_j$, along with the dimensions of these components. Consequently, by identifying the algebraic structure of a DLA, one can ascertain whether BPs will be present. In particular, if the DLA decomposes into at most a polynomial number of simple components and all of them have exponential dimension, then it will lead to BPs.

While the mathematical relationship between DLAs and BPs is enlightening, the practical application of this connection requires the explicit computation of various DLA properties, such as the dimension and basis of a DLA, its decomposition into a center and simple components, and a set of basis vectors for each simple component. A basis for a DLA can be computed algorithmically, although the time required can be exponential in the number of qubits. However, determining the decomposition and component bases presents a significantly greater challenge.

In this work, we focus on the DLA of one of the most studied variational quantum algorithms, namely the Quantum Approximate Optimization Algorithm (QAOA) [FGG14] for the MaxCut problem on graphs. Given an unweighted graph $G = (V, E)$, the QAOA-MaxCut circuit corresponds to the layered variational circuit

$$U(\gamma, \beta) = e^{-i\beta_L H_m} e^{-i\gamma_L H_p} \dots e^{-i\beta_1 H_m} e^{-i\gamma_1 H_p}$$

where L is a positive integer, $\gamma = (\gamma_1, \dots, \gamma_L)$ and $\beta = (\beta_1, \dots, \beta_L)$ with $\beta_j, \gamma_j \in \mathbb{R}$, and the mixer and problem Hamiltonians are

$$H_m \triangleq \sum_{u \in V} X_u, \quad H_p \triangleq \sum_{(u,v) \in E} Z_u Z_v,$$

respectively, where X_u, Z_u denote the Pauli X and Z operators acting on qubit u . The corresponding DLA $\mathfrak{g}$ is generated by the set $\{H_m, H_p\}$.

A variant of QAOA called multi-angle QAOA [HLO⁺22] was developed to enhance the approximation ratio for the MaxCut problem while reducing circuit depth. While the standard

algorithm assigns a single parameter (β_j) to each H_m , and another (γ_j) to each H_p , in multi-angle QAOA each X_u and $Z_u Z_v$ term is tunable by its own independent parameter. Thus, the *multi-angle DLA* $\mathfrak{g}_{\text{ma}}$ is generated by the set $\{X_u : u \in V\} \cup \{Z_u Z_v : (u, v) \in E\}$. It is always the case that $\mathfrak{g} \subseteq \mathfrak{g}_{\text{ma}}$ and if equality holds we say that the standard DLA $\mathfrak{g}$ is *free*. In contrast to the standard DLA case, where the DLA generators are linear combinations of Pauli strings, each generator of the multi-angle DLA is an individual Pauli string. This facilitates the analysis, and a full characterization of multi-angle DLAs on general connected graphs has been obtained [KLF⁺25, KWKB24].

The standard QAOA-MaxCut DLAs have proven more challenging to analyze, and to date are well-understood only for some symmetric cases, such as paths [KLF⁺25], cycles [ASYZ26, DI25], and complete graphs [ASYZ26]. However, the situation for general graphs is not well understood. Numerical investigations on Erdős-Rényi random graphs [LCS⁺22] suggest that typical instances exhibit large-dimensional DLAs and are prone to BPs.

In this work, we investigate the standard QAOA-MaxCut DLA for both unweighted graphs, as described above, as well as weighted graphs $G = (V, E, \mathbf{r})$, where $\mathbf{r} = \{r_{uv} \in \mathbb{R} : (u, v) \in E\}$ denotes the set of weights on edges.

We first give a sufficient condition for the DLA of any weighted graph to be free.

Theorem 1. *Let $G = (V, E, \mathbf{r})$ be a weighted graph such that*

1. $r_{uv} \neq 0$ for any $(u, v) \in E$, and
2. for any family of sign functions $\{s_x : \mathcal{N}(x) \rightarrow \{\pm 1\}\}_{x \in V}$, and any two vertices $u, v \in V$,

$$\sum_{w \in \mathcal{N}(u)} s_u(w) r_{uw} \neq \sum_{w' \in \mathcal{N}(v)} s_v(w') r_{vw'}. \quad (1)$$

Then $\mathfrak{g} = \mathfrak{g}_{\text{ma}}$.

The conditions on the edge weights r_{uv} in the above theorem are satisfied with probability one for connected graphs on $n \geq 3$ vertices with weights randomly sampled from continuous distributions. By combining this with a known classification of all multi-angle QAOA-MaxCut DLAs, we obtain the following:

Theorem 2. *Let $G = (V, E, \mathbf{r})$ be a connected graph, and not a path or cycle. If the weights are drawn from a continuous distribution, then, with probability 1, the following statements hold for QAOA-MaxCut.*

1. $\mathfrak{g} = \mathfrak{g}_{\text{ma}}$.
2. $\mathfrak{g}$ is the direct sum of either one or two simple Lie algebras, each of dimension $\Theta(4^n)$.
3. $\text{Var}_{\boldsymbol{\theta}}[\ell(\rho, O; \boldsymbol{\theta})] = O(1/2^n)$, and barren plateaus are therefore present.

For unweighted graphs — corresponding to the standard QAOA-MaxCut setting — we prove that, asymptotically, all but an exponentially small fraction of graphs have exponentially large DLA dimension, confirming the numerical observations of [LCS⁺22]. Combined with the fact that such random graphs are connected with high probability, this again implies that $\mathfrak{g}$ has exponentially large components, and the corresponding QAOA-MaxCut algorithms will be susceptible to barren plateaus. More precisely, we consider Erdős-Rényi random graphs $G \sim G(n, p)$, where graphs on n vertices are constructed by adding each possible edge between pairs of vertices independently with probability p . When $p = 1/2$ we obtain the following result.

Theorem 3. *Let $G \sim G(n, \frac{1}{2})$ be an Erdős-Rényi random graph. Then with probability at least $1 - \exp(-\Omega(n))$, the following statements hold for QAOA-MaxCut.*

1. $\mathfrak{g} = \mathfrak{g}_{\text{ma}}$.
2. $\mathfrak{g}$ is the direct sum of either one or two simple Lie algebras, each of dimension $\Theta(4^n)$.
3. $\text{Var}_{\theta}[\ell(\rho, O; \theta)] = O(1/2^n)$, and barren plateaus are therefore present.

The same is true if we draw G uniformly at random from all non-isomorphic graphs on n vertices.

Our proof of Theorem 3 is based on a novel method for converting complicated Lie algebraic problems into amenable graph-theoretic problems. This approach forms the basis for a new algorithm that computes DLAs orders of magnitude faster than previous methods. We apply this algorithm to the MQLib benchmark suite covering over 3,500 graph instances with up to 53,130 vertices, and find that at least 75% of the instances possess a DLA of dimension at least 2^{128} .

In addition, we give explicit constructions for families of graphs whose DLAs are free, as well as constructive methods for embedding graphs with free DLAs in larger graphs such that the DLAs of the larger graphs are also free. In particular, we consider so-called asymmetric subdivisions of odd graphs, extended ladder graphs, and grid+1 graphs and show that their corresponding DLAs are all free.

Theorem 4. *If $G = (V, E)$ is (i) an asymmetrically subdivided odd graph, (ii) an extended ladder graph, or (iii) a grid+1 graph, then $\mathfrak{g} = \mathfrak{g}_{\text{ma}}$.*

As a consequence, it follows that for every $n \geq 7$, there exists at least one n -vertex graph whose DLA is free.

Since the cost functions are the same for standard QAOA and multi-angle QAOA, for any graph from the three families in Theorem 4, $\text{Var}_{\theta}[\ell(\rho, O; \theta)]$ should be the same whether the standard QAOA or multi-angle QAOA ansatz is used, provided that the circuit depths are sufficiently deep that the characterization of the variance in terms of DLAs is valid.

We further give a polynomial-time reduction from any connected unweighted graph to an asymmetrically subdivided odd graph which is an equivalent MaxCut instance. The freeness of the corresponding DLA shows that MaxCut instances with exponentially large DLAs are ubiquitous, and include those that are efficiently solvable classically.

Theorem 5. *For any connected graph $G = (V, E)$, there is an asymmetrically subdivided odd graph $G' = (V', E')$, with $|V'|, |E'| \in O(|E|^2)$, such that*

1. $\text{MaxCut}(G') - \text{MaxCut}(G) = |V'| - |V|$, and there is an efficient transformation between the MaxCut solutions of G and G' .
2. $\mathfrak{g}' = \mathfrak{g}'_{\text{ma}}$, where $\mathfrak{g}'$ is the DLA corresponding to G' .

References

- [ASYZ26] Jonathan Allcock, Miklos Santha, Pei Yuan, and Shengyu Zhang. On the dynamical Lie algebras of quantum approximate optimization algorithms. *Quantum*, 10:2119, May 2026. DOI: [10.22331/q-2026-05-29-2119](https://doi.org/10.22331/q-2026-05-29-2119).
- [DI25] Domenico D'Alessandro and Yasemin Isik. Controllability of the periodic quantum Ising spin chain and the Onsager algebra. *Journal of Physics A: Mathematical and Theoretical*, 58(11):115202, March 2025. DOI: [10.1088/1751-8121/adbdb0](https://doi.org/10.1088/1751-8121/adbdb0).
- [FGG14] Edward Farhi, Jeffrey Goldstone, and Sam Gutmann. A quantum approximate optimization algorithm, 2014. arXiv preprint [arXiv:1411.4028](https://arxiv.org/abs/1411.4028) [quant-ph].

- [FHC⁺24] Enrico Fontana, Dylan Herman, Shouvanik Chakrabarti, Niraj Kumar, Romina Yalovetzky, Jamie Heredge, Shree Hari Sureshababu, and Marco Pistoia. Characterizing barren plateaus in quantum ansätze with the adjoint representation. *Nature Communications*, 15(1):7171, August 2024. DOI: [10.1038/s41467-024-49910-w](https://doi.org/10.1038/s41467-024-49910-w).
- [HLO⁺22] Rebekah Herrman, Phillip C Lotshaw, James Ostrowski, Travis S Humble, and George Siopsis. Multi-angle quantum approximate optimization algorithm. *Scientific Reports*, 12(1):6781, April 2022. DOI: [10.1038/s41598-022-10555-8](https://doi.org/10.1038/s41598-022-10555-8).
- [KLF⁺25] Sujay Kazi, Martín Larocca, Marco Farinati, Patrick J. Coles, M. Cerezo, and Robert Zeier. Analyzing the quantum approximate optimization algorithm: Ansätze, symmetries, and Lie algebras. *PRX Quantum*, 6:040345, Nov 2025. DOI: [10.1103/yfwq-yqmk](https://doi.org/10.1103/yfwq-yqmk).
- [KWKB24] Efehan Kökcü, Roeland Wiersema, Alexander F. Kemper, and Bojko N. Bakalov. Classification of dynamical Lie algebras generated by spin interactions on undirected graphs, 2024. arXiv preprint [arXiv:2409.19797](https://arxiv.org/abs/2409.19797) [quant-ph].
- [LCS⁺22] Martin Larocca, Piotr Czarnik, Kunal Sharma, Gopikrishnan Muraleedharan, Patrick J. Coles, and M. Cerezo. Diagnosing barren plateaus with tools from quantum optimal control. *Quantum*, 6:824, September 2022. DOI: [10.22331/q-2022-09-29-824](https://doi.org/10.22331/q-2022-09-29-824).
- [MBS⁺18] Jarrod R McClean, Sergio Boixo, Vadim N Smelyanskiy, Ryan Babbush, and Hartmut Neven. Barren plateaus in quantum neural network training landscapes. *Nature Communications*, 9(1):4812, 2018. DOI: [10.1038/s41467-018-07090-4](https://doi.org/10.1038/s41467-018-07090-4).
- [RBS⁺24] Michael Ragone, Bojko N. Bakalov, Frédéric Sauvage, Alexander F. Kemper, Carlos Ortiz Marrero, Martín Larocca, and M. Cerezo. A Lie algebraic theory of barren plateaus for deep parameterized quantum circuits. *Nature Communications*, 15(1):7172, August 2024. DOI: [10.1038/s41467-024-49909-3](https://doi.org/10.1038/s41467-024-49909-3).