

Reversing Unknown Hamiltonian Evolutions through Algebraic Structure

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Abstract. Coherent inverse access is needed in amplitude amplification, time-reversal tests, and scrambling probes, but exact black-box inversion of an unknown d -dimensional unitary has quadratic query cost. We ask what changes when the oracle is a Hamiltonian evolution $U(x) = \exp(i \sum_j x_j H_j)$ with known generators and unknown couplings. In a compiled-structure query model, fixed gates may depend on the known generator algebra while only coherent calls to the unknown forward evolution are counted. For one Hamiltonian with known eigenspaces, $K - 1$ queries suffice for K distinct eigenvalues, and the optimum is an additive sumset condition on the distinct spectrum. For multi-parameter families, Wedderburn reduction removes passive multiplicity wires, and local active-block inverses can be recombined by synchronizing their query lengths and residual scalar phases. Determinant, trace-vector, and automatic-completion constructions give explicit exact protocols. Exchange, hopping, and collective-spin examples show query budgets controlled by active representation data rather than by the full many-body Hilbert space.

Keywords: Quantum algorithms, Hamiltonian dynamics, unitary inversion, higher-order transformations, representation theory

Introduction. Many quantum routines need a process to be run both forward and backward. Amplitude amplification and estimation use inverse calls to obtain their quadratic speedups, while Loschmidt echoes and OTOCs compare forward and backward evolution to probe scrambling [1, 2]. Operational time reversal treats the inverse as a resource in its own right [3]. These tasks fit naturally into the language of higher-order transformations, where a circuit transforms an unknown process instead of merely applying it [4–8].

For a completely unknown unitary, exact inversion is expensive. Universal protocols began with probabilistic constructions and adaptive or indefinite-causal-order separations [9–11]. Deterministic exact inversion is known for qubits and for arbitrary finite dimension, but the generic d -dimensional query cost scales quadratically in d [12, 13]. Worst-case lower bounds and approximation barriers show that this is not just an artifact of the known algorithms [14, 15].

Hamiltonian experiments are usually less mysterious than an arbitrary unitary box. The graph, symmetries, local terms, and controls are often fixed by design; the hidden data are the realized couplings. We therefore study families

$$U(x) = \exp\left(i \sum_{j=1}^p x_j H_j\right), \quad x \in \mathbb{R}^p,$$

where the Hermitian generators H_j are known and x is not. The compiler may use any fixed gate depending on the known family, but the only x -dependent operation it may call is the forward oracle $U(x)$. This is a query model. It says how many uses of the unknown evolution are needed; it does not claim that the compiled basis changes or controlled gates are free to build.

Overview of results. A clean q -query comb succeeds if, for every hidden parameter x , it outputs $e^{i\phi(x)}U(x)^\dagger$ and returns its ancillas cleanly. Within this model, the paper proves the following query-level statements.

1. **Spectral routing.** If one Hamiltonian has a known eigenspace decomposition and K distinct eigenvalues, $K - 1$ forward calls always implement the inverse, independent of eigenspace multiplicities.
2. **The exact scalar optimum.** The minimum query number is the smallest q for which a shifted copy of the distinct spectrum lies in the q -fold sumset. The answer is an arithmetic invariant of the spectrum, not of the ambient Hilbert-space dimension.
3. **Generated algebras replace moving eigenvectors.** For several noncommuting components, the useful fixed object is the $*$ -algebra generated by $H_1, \dots, H_p$. Its Wedderburn decomposition separates passive multiplicity spaces from active blocks; the passive wires do not change the exact inversion query number.
4. **Blockwise synchronization.** Local active-block inverses can be glued together only after their query lengths and scalar phases agree. This gives a modular compiler, with determinant, trace-vector, and automatic completion constructions as reusable exact instantiations.

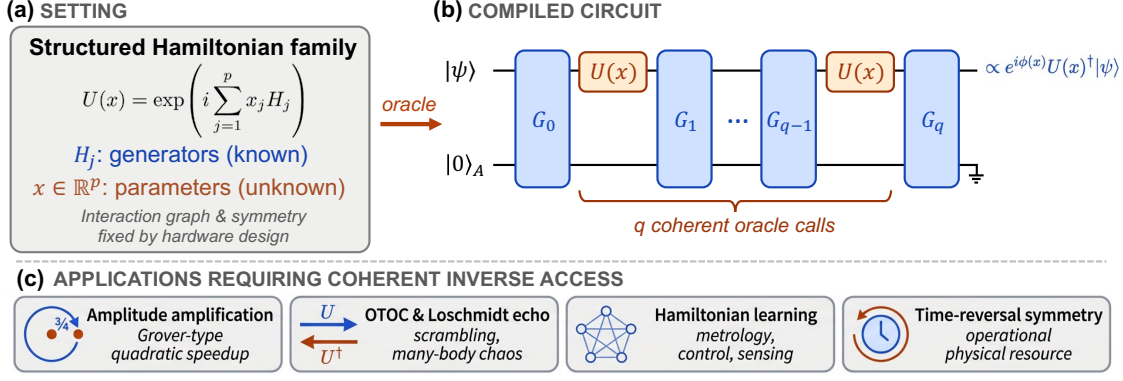


Figure 1: **Compiled-structure unitary inversion.** Known generators and unknown parameters define a structured oracle $U(x)$. Fixed gates may use the known algebra, while the query count charges only coherent calls to the unknown forward evolution. The output is coherent inverse access, up to a global phase, uniformly in x .

A single Hamiltonian. Start with $U_t = e^{itH}$, where the eigenspaces of H are known but the time t is not. A fixed encoder stores the multiplicity label in an ancilla and routes a representative vector from each eigenspace through the other eigenspaces. The accumulated phase is the missing inverse phase, up to a common factor.

Theorem 1. *If H has K distinct eigenvalues, there is a clean deterministic exact protocol using $K - 1$ calls to U_t that implements $e^{i\beta t} U_t^\dagger$, with $\beta = \sum_j \lambda_j$, for all $t \in \mathbb{R}$.*

This protocol is universal for spectra of size K , but it is not always optimal. Let Λ be the set of distinct eigenvalues and $\Sigma_q(\Lambda)$ its q -fold sumset.

Theorem 2. *In the clean deterministic exact fixed-eigenbasis model, the minimum query number is*

$$\kappa(H) = \min\{q \geq 0 : \exists c \in \mathbb{R}, c - \Lambda \subseteq \Sigma_q(\Lambda)\}.$$

It depends only on the distinct eigenvalues, not on eigenspace multiplicities.

The lower bound is a Fourier-support argument: a q -query comb can only produce frequencies in $\Sigma_q(\Lambda)$, while exact inversion forces the global scalar phase to be a single Fourier mode. The upper bound is an explicit routing circuit from any feasible sumset witness. For example, $\Lambda = \{-5, -4, 1, 3, 5\}$ has $\kappa(H) = 3 < K - 1 = 4$.

Families of Hamiltonians. With several noncommuting generators, the eigenvectors of $\sum_j x_j H_j$ move with x , so fixed spectral routing no longer applies. The fixed structure is instead the finite-dimensional algebra $\mathcal{A} = \ast\text{alg}(I, H_1, \dots, H_p)$. After a Wedderburn basis change,

$$\mathcal{H}_S \simeq \bigoplus_{\alpha} (M_{\alpha} \otimes V_{\alpha}), \quad U(x) \simeq \bigoplus_{\alpha} (I_{M_{\alpha}} \otimes U_{\alpha}(x)).$$

The M_{α} wires only count how many copies of the same active representation appear.

Theorem 3. *For every q , a clean exact q -query inverse for $U(x)$ exists if and only if one exists for the reduced oracle $U_{\text{red}}(x) = \bigoplus_{\alpha} U_{\alpha}(x)$ on $\bigoplus_{\alpha} V_{\alpha}$.*

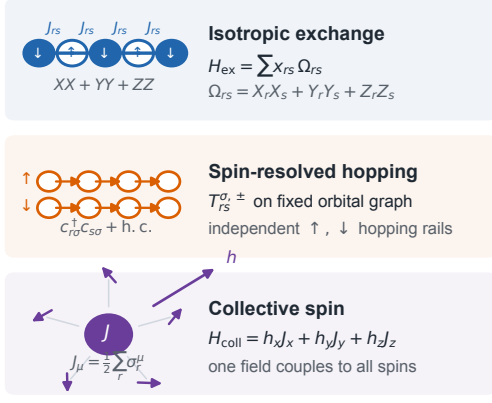
Thus multiplicities are not a query resource. They can be split off before the oracle call and restored afterward by fixed wirings.

Blockwise synchronization. After reduction, different active blocks must still be recombined coherently. A local inverter on block α may output $\rho_{\alpha}(x) U_{\alpha}(x)^\dagger$; the phase $\rho_{\alpha}(x)$ is harmless inside one block but becomes a relative phase across a direct sum. The compiler therefore pads branches and adds scalar phase gadgets until every branch has the same query length and the same residual scalar phase.

Theorem 4. *Suppose each block has a local clean inverter of cost ℓ_{α} and residual phase $\rho_{\alpha}(x)$, and suppose available scalar gadgets can make all branches share a common query length q and phase $\zeta(x)$. Then the full family has a clean exact q -query inverse implementing $\zeta(x) U(x)^\dagger$.*

The theorem is deliberately modular. The local block primitive may come from a universal unitary inverter or from extra structure in that block. Even when no closed formula for ρ_{α} is known, a closed auxiliary rail turns the local inverter itself into a scalar gadget, giving an automatic completion budget $\sum_{\alpha} (\ell_{\alpha} + 1) - 1$.

a Physical models



b

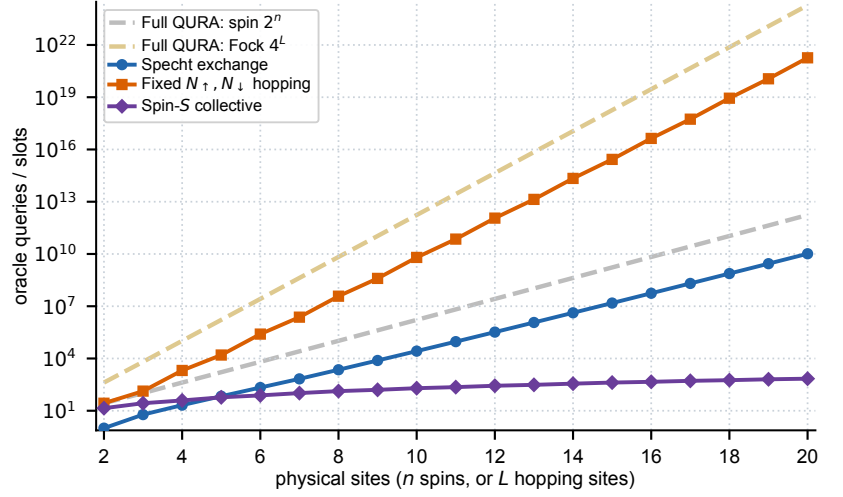


Figure 2: Known Hamiltonian structure shifts the query budget from the full space to the active blocks. (a) We compare three standard many-body settings: connected isotropic exchange, spin-resolved hopping on a fixed orbital graph, and collective spin. (b) If these evolutions are treated as arbitrary unitaries, QURA pays for the full spin or spinful Fock space (dashed). Using the known generators first reduces the oracle to its active Wedderburn blocks, giving the exact query upper bounds shown by the solid curves. These are query-level costs only: they count coherent calls to the unknown forward evolution, not the fixed compiled gates used for reduction, routing, or recombination.

Universal determinant inverters give the canonical construction: on a d_α -dimensional block they return $(\det U_\alpha)^{s_\alpha} U_\alpha^\dagger$, and filled-shell gadgets synchronize the determinant phases. Trace-vector relations can reduce this determinant inventory; if every active generator is traceless, determinant phases vanish and the problem collapses to query-length padding.

Physical illustrations. Figure 2 instantiates the compiler with the QURA universal block inverter [13], whose d -dimensional cost is

$$q_{\text{QURA}}(d) = d \left\lceil \frac{\pi}{2 \arcsin(1/d)} \right\rceil - 1.$$

The plotted curves are certified exact upper bounds in the compiled-structure query model. They should not be read as end-to-end hardware costs.

The examples are deliberately ordinary. In isotropic exchange, the unknown parameters are Heisenberg couplings on a known connected spin graph, as in exchange-based spin-qubit models [16]. The swap relation turns this family into the symmetric-group representation [17]. Thus the active blocks are two-row Specht modules, and their nonzero traces require determinant synchronization.

Spin-resolved hopping is the kinetic part of a spinful fermion model on a known orbital graph [18, 19]. Fixed $(N_\uparrow, N_\downarrow)$ sectors have dimensions $\binom{L}{N_\uparrow} \binom{L}{N_\downarrow}$. Pure hopping is traceless on each sector, so the compiler only pads all sectors to the largest block.

Collective spin is a different limit: all spins couple to the same unknown field [20, 21]. Schur-Weyl duality removes the passive multiplicities and leaves spin- S irreducible representations. The largest active block has dimension $n + 1$, even though the ambient spin space has dimension 2^n . Together, the three cases show the same mechanism in physical terms: pay for active representation size and phase synchronization, not for passive copies.

Conclusion. The black-box barrier for exact inversion is real, but it is a barrier for generic unitaries. When the oracle is known to come from a Hamiltonian family, the relevant query dimension is the active representation through which the unknown evolution flows. The fixed-eigenbasis case makes this exact through a sumset optimum; the noncommuting case replaces spectral labels by Wedderburn active blocks and phase synchronization. Turning these query savings into laboratory savings still requires compiling the fixed gates efficiently. That synthesis problem is separate, and the present work keeps it separate on purpose.

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