

Extended abstract — Lower bounding the MaxCut of high girth 3-regular graphs using the QAOA

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In this paper we study MaxCut on 3-regular graphs with large girth. The girth of a graph is the length of its shortest cycle. We define M_g as the infimum of the maximum cut fraction over all 3-regular graphs with girth g . Our work makes progress on two fronts: establishing new mathematical lower bounds on M_g , and demonstrating an exponential quantum algorithmic advantage for the task of finding cuts that meet these bounds, when compared to currently known classical algorithms guaranteed to achieve a cut of that size. Both results are derived from an analysis of the performance of the Quantum Approximate Optimization Algorithm [1].

The Quantum Approximate Optimization Algorithm (QAOA) is a quantum algorithm for finding approximate solutions to combinatorial optimization problems. We seek to optimize an objective function C over bit strings, which is usually a sum over terms that each involves a few bits. The QAOA depends on an integer parameter $p \geq 1$ and produces quantum states of the form

$$|\gamma, \beta\rangle = U(B, \beta_p)U(C, \gamma_p) \cdots U(B, \beta_1)U(C, \gamma_1)|s\rangle. \quad (1)$$

Here $|s\rangle$ is the uniform superposition over computational basis states, $U(C, \gamma) = e^{-i\gamma C}$ is diagonal in the computational basis, and $U(B, \beta) = e^{-i\beta B}$ where $B = \sum_j X_j$ is the sum of single-qubit Pauli X operators. The angles $\gamma = (\gamma_1, \dots, \gamma_p)$ and $\beta = (\beta_1, \dots, \beta_p)$ are classical parameters which specify the QAOA state. For MaxCut, given a graph with vertices V and edges E , the cost function operator counts edges crossing between two subsets in a partition,

$$C = \sum_{(j,k) \in E} \frac{1 - Z_j Z_k}{2}. \quad (2)$$

When measured in the computational basis, the state $|\gamma, \beta\rangle$ produces a bit string with an associated cost function value equal to the cut size and whose quantum expectation is $F_p(\gamma, \beta) = \langle \gamma, \beta | C | \gamma, \beta \rangle$. The angles (γ, β) are optimized to maximize the expectation value $F_p(\gamma, \beta)$. The quantum circuit depth grows with p , and the performance at optimal parameters can only improve with larger p .

For graphs with high girth ($g \geq 2p + 2$), the causal cone of any single edge in the QAOA circuit is a tree, as seen in Fig. 1. For d -regular graphs, this tree is isomorphic for every edge, meaning the expected cut fraction $\langle \gamma, \beta | C | \gamma, \beta \rangle / |E|$ can be calculated by analyzing any single edge neighborhood,

$$\langle \gamma, \beta | C | \gamma, \beta \rangle / |E| = (1 - \langle \gamma, \beta | Z_i Z_j | \gamma, \beta \rangle) / 2 \equiv c_{\text{edge}}(\gamma, \beta), \quad (3)$$

where $|E|$ is the total number of edges and (i, j) is any pair of neighbors. For fixed depth p , for any parameters γ, β , one then obtains the rigorous lower bound $M_g \geq c_{\text{edge}}(\gamma, \beta)$ for $g \geq 2p + 2$. Maximizing over γ, β then yields an optimized lower bound, $M_g \geq \tilde{c}_{\text{edge}}(p)$.

Contribution 1 — New lower bounds on the maximum cut. Our primary mathematical result is a new set of lower bounds on M_g for 3-regular graphs, derived from the classical, exact numerical evaluation of the QAOA's performance.

As shown in Fig. 2 (left), our results improve upon the best previously known bounds for all $g \geq 16$, which were established by Thompson, Parekh, and Marwaha (TPM) [5]. Notably, the asymptotic TPM bound as $g \rightarrow \infty$ is 0.8918, a value our QAOA-derived bound surpasses at $g \geq 32$ ($p \geq 15$). By pushing the analysis to a circuit depth of $p = 17$, we establish a bound for graphs with girth $g \geq 36$ of $M_{36} \geq 0.8971$. This demonstrates

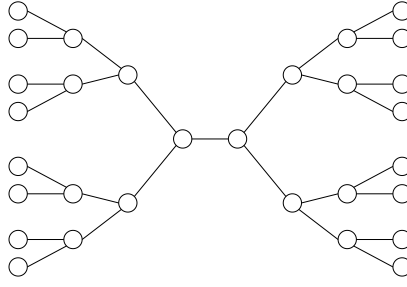


Figure 1: In a regular graph, the neighborhood of each edge has a regular tree structure, whenever the diameter of the neighborhood is smaller than the girth of the graph. To calculate the expected cut fraction $c_{\text{edge}}(\gamma, \beta)$, we use the tensor network representing the quantum circuit defined over the tree. We evaluate the network using a new method to exploit symmetry, which reduces the complexity compared to previous works [2, 3, 4]; see Contribution 3.

the power of analyzing quantum algorithms to uncover new graph-theoretic results. The existence of a cut of this size is guaranteed by the analysis of the performance of the QAOA and no quantum computer is needed to establish the bound.

Interestingly, while our QAOA large-girth cut fraction guarantees apply in the worst case, they also *apply to typical large 3-regular graphs*. This is because for typical large 3-regular graphs almost all edge neighborhoods of fixed size are trees. The best value of the cut fraction achieved by our QAOA analysis (0.8971 at $p = 17$) is therefore both (1) a worst-case guarantee when $g \geq 36$ and (2) an average-case guarantee for large random 3-regular graphs. In the limit $g \rightarrow \infty$ it is known that M_g is lower bounded by 0.912 [7, 6]. If the QAOA

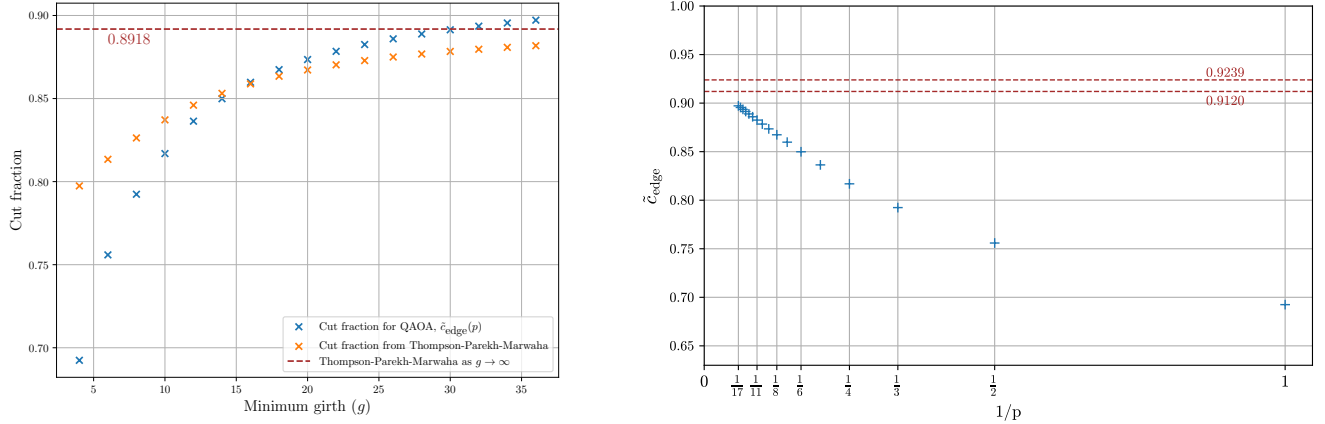


Figure 2: **Left:** Our new lower bounds on M_g (blue crosses) for 3-regular graphs of minimum girth g , obtained by analyzing the QAOA, compared with the bounds of Thompson–Parekh–Marwaha (TPM; orange crosses). Our bound surpasses TPM for $g \geq 16$ and exceeds the TPM asymptote by $g \geq 32$. **Right:** Optimized cut fraction $\tilde{c}_{\text{edge}}(p)$ versus $1/p$. The value 0.912 is the lower bound on $\lim_{g \rightarrow \infty} M_g$ given by Refs. [6, 7]. If $\tilde{c}_{\text{edge}}$ were to exceed this value at larger p , it would provide a new lower bound on $\lim_{g \rightarrow \infty} M_g$. The value 0.9239 is the upper bound on the expected cut fraction of large random 3-regular graphs given in [8], and $\tilde{c}_{\text{edge}}$ cannot exceed this value.

surpasses this value in the limit of large g (see Fig. 2 right), it would provide a new lower bound on $\lim_{g \rightarrow \infty} M_g$. Furthermore, the best known upper bound on the cut fraction is 0.9239 [8].

Contribution 2 — Exponential quantum advantage. While the existence of these large cuts is established by our methods, finding them requires running an algorithm, either classical or quantum. Our second key contribution is to show that the QAOA, when run on a quantum computer, provides an exponential

advantage over all known classical algorithms *with performance guarantees* for obtaining cuts of size equal to the proven lower bounds. For a constant depth p , the QAOA circuit has $O(|E|)$ gates. By preparing the state $|\tilde{\gamma}, \tilde{\beta}\rangle$ with pre-computed optimized angles and measuring in the computational basis, we can sample a cut. The expected size of this cut is $|E| \cdot \tilde{c}_{\text{edge}}(p)$. With $O(|E|)$ samples, one can find a cut of at least this size with high probability.

For $g \geq 16$ no classical algorithm known to us is guaranteed to find a cut of this size in subexponential time. The best classical algorithms with such a guarantee, such as brute-force search or more sophisticated variants [9], require time that scales exponentially with the size of the graph. Other classical algorithms, such as Goemans-Williamson (GW) [10], will outperform the QAOA in some instances and come equipped with an approximation ratio guarantee, but they have an inferior guarantee for cut fraction in the large-girth regime we consider.

Contribution 3 — Improved analytical and numerical techniques. The results presented above rely on our third contribution: an improvement in the techniques used to compute the cut fraction of Eq. (3). The expectation value of $\langle \gamma, \beta | Z_i Z_j | \gamma, \beta \rangle$ can be calculated by contracting a tensor network representing the quantum circuit in the causal cone of an edge in the graph, supported over a tree subgraph (Fig. 1). For a d -regular graph ($d \geq 3$), the number of qubits in the subgraph grows exponentially with the depth p .

Previous work has analyzed the performance of the QAOA on 3-regular graphs up to $p = 11$ in the context of the fixed-angle conjecture [3] via an optimized tensor network contraction over the exponentially large circuit, which is difficult to execute at higher p , and whose complexity suffers from a steep dependency in d . Other work developed a recursive formula for the evaluation of c_{edge} with complexity $O(16^p)$ [2, 4].

To obtain better performance guarantees at high girth than those of the classical algorithm in Ref. [5], we require a more efficient method for evaluating the QAOA performance; the methods developed in Refs. [2, 3, 4] are practically intractable for this purpose. To this end, we exploit the symmetries of the regular tree structure of the support of the circuit. Instead of working with the exponentially large tensor network, we contract a single branch inwards towards the root, raising the resulting tensor's entries to the power of $(d - 1)$ at each level before proceeding to later contractions. This technique reduces the computational complexity to a scaling $O(4^p)$, which is independent of the graph's degree d , and allows us to perform exact numerical calculations up to $p = 17$ for any $d \geq 3$.

Interestingly, we see that the parameters which optimize $c_{\text{edge}}(\gamma, \beta)$ have a regular structure as indicated in Fig. 3. This helps in finding optimal parameters as we increase the depth p .

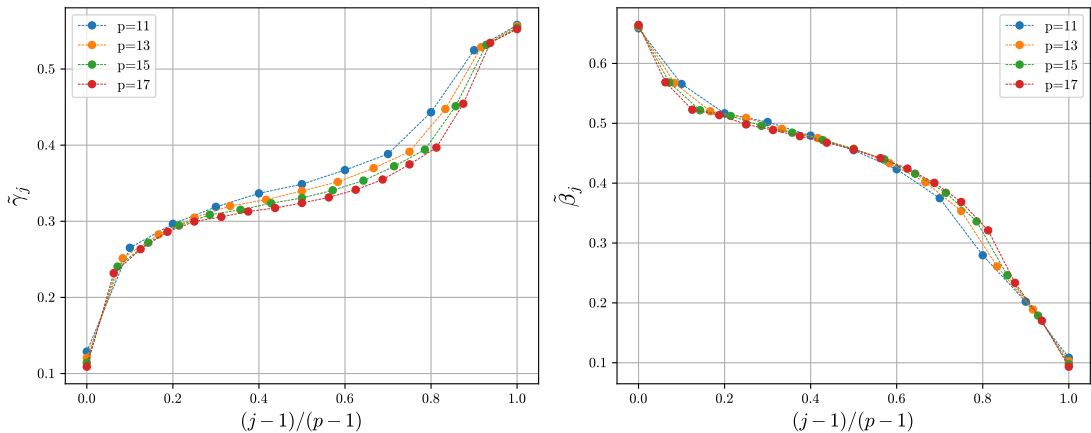


Figure 3: For four different p values we plot the optimized parameters $\tilde{\gamma}_j$ (left) and $\tilde{\beta}_j$ (right), each as a function of the layer index $j = 1, \dots, p$. The data suggests the optimal parameters might approach fixed curves as p increases. In particular, the parameters do not seem to vanish as p increases, indicating the QAOA is *not* merely reproducing a discretized adiabatic algorithm.

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