

# Fourier analysis of quantum neural network with non-linear data embedding

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**Abstract.** Fourier analysis [1–5] has become a crucial tool for understanding the expressivity of Variational Quantum Circuit (VQC) models [7–10], as well as an important indicator of barren plateaus (BP) [11]. While existing literature has only studied angle-embedded VQCs in a noiseless environment [1–5], here we [6] develop the Fourier analysis of VQCs with non-linear data embedding, with particular focus on amplitude embedding [12], which provides a naturally compact encoding scheme. In Section 3, we present our main result that, by assuming that the parametrised unitaries form a 2-design with respect to the full unitary group, the mean of the Fourier coefficients is concentrated at zero, while the variance scales at an exponentially decaying order with respect to the multi-dimensional frequency magnitude via Weingarten Calculus. When a noise channel with unitary Kraus operators and probabilities  $\{p_k\}$  is taken into account, we show that the variance is further suppressed by a factor  $(\sum_k p_k^2)^Q < 1$ , where  $Q$  is the number of channel instances applied. On top of this, we demonstrate and validate the analytical results through simulations, both noiseless and noisy. Beyond the length of this extended abstract, this work also investigates how differences in the domain of amplitudes can lead to a distinct expressivity of the zero-frequency Fourier coefficient, as well as demonstrating its utility in fitting target functions with non-integer frequencies that are forbidden in vanilla angle-embedded models. Our results establish a rigorous Fourier framework for amplitude-encoded VQCs, offering both theoretical guarantees on expressivity, hence trainability scaling in the frequency domain, as well as practical simulations for deployment on noisy quantum devices.

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## 1 Fourier analysis with angle-embedded variational circuit

A VQC model [9, 10], is in general, a parametrized function  $f : \mathcal{X} \times \Theta \rightarrow \mathcal{Y}$  defined as the expectation value of some observable evaluated on a parametrized quantum circuits (PQC), such that

$$f_{\theta}(\mathbf{x}) = \langle 0 | U(\mathbf{x}, \theta)^\dagger O U(\mathbf{x}, \theta) | 0 \rangle \quad (1)$$

where  $\mathcal{X}, \mathcal{Y}, \Theta$  denote the feature, target, and parameter spaces, respectively.  $U(\mathbf{x}, \theta)$  is a unitary as a function of the input feature vector  $\mathbf{x} = (x_1, x_2, \dots, x_d)^T \in \mathcal{X}$  of dimension  $d$  and parameters  $\theta = (\theta_1, \theta_2, \dots, \theta_M)^T \in \Theta$ . The VQC model,  $f$ , can also be expressed as a truncated Fourier series in the frequency domain,

$$f_{\theta}(\mathbf{x}) = \sum_{\omega \in \Omega} c_{\omega}(\theta) e^{-i\omega \cdot \mathbf{x}}, \quad (2)$$

where  $\omega = (\omega_1, \omega_2, \dots, \omega_d) \in \Omega$  are the frequencies comprising the Fourier spectrum,  $c_{\omega}$  are the corresponding Fourier coefficients. In the well-studied case of angle embedding, studies [1, 3] have shown that the spectrum for angle-embedded VQC is simply the possible differences between the eigenvalue sums of the encoding Hamiltonian:

$$\Omega = \left\{ \omega = (\omega_1, \omega_2, \dots, \omega_d) \mid \omega_k = A_{\mathbf{J}}^{(k)} - A_{\mathbf{J}'}^{(k)} \in \{-L, -L+1, \dots, L\}, \mathbf{J}, \mathbf{J}' \in \llbracket 1, 2^n \rrbracket^L \right\}, \quad (3)$$

where  $A_{\mathbf{J}}^{(k)} = \sum_{l=1}^L \lambda_{j_l}^{(k,l)}$ . Moving one step further, we show that for multidimensional input features,

$$\text{Var}_{\theta}(c_{\omega}(\theta)) \leq O\left(\alpha \frac{|\tilde{R}(\omega)|}{2^n}\right) \leq O\left(\prod_{k=1}^d \binom{2L}{L - |\omega_k|}\right), \quad (4)$$

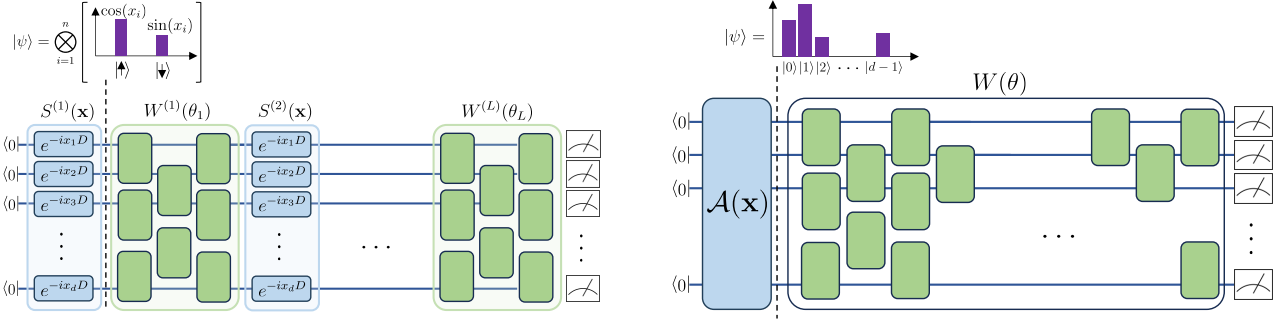


Fig. 1: Left panel: The structure of the circuit  $U(\mathbf{x}_i, \boldsymbol{\theta})$  when input features are encoded via the angle-embedding method with data reuploading protocol, given that a fixed encoding Hamiltonian  $D$  is used throughout the circuit. Right panel: The structure of the variational circuit  $U(\mathbf{x}_i, \boldsymbol{\theta})$  when input features are encoded via the amplitude-embedding method. Unlike angle-embedding circuits, the encoding layers and variational layers are distinctively separated.

where  $|\tilde{R}(\boldsymbol{\omega})|$  is directly proportional to the number of eigenvalues that can produce the target frequency  $\boldsymbol{\omega}$ .

## 2 Fourier analysis with amplitude-embedded variational circuit

On the other hand, the structure of a VQC with amplitude embedding first encodes the input features into the amplitudes of the computational basis states, where

$$\mathcal{A}(\mathbf{x}) |0\rangle^{\otimes n} = |\mathbf{x}\rangle = \frac{1}{\|\mathbf{x}\|_2} \sum_{i=0}^{2^n-1} x_i |i\rangle. \quad (5)$$

Hence, the Fourier coefficients prescribed by Equation (2) must take the following form:

$$c_{\boldsymbol{\omega}}(\boldsymbol{\theta}) = \sum_{i, i' \in [d]} \int_D d^{(d)} V \frac{x_i x_{i'} W(\boldsymbol{\theta})_{i'j'}^\dagger O_{j'j} W(\boldsymbol{\theta})_{ji}}{(2\pi)^d \|\mathbf{x}\|_2^2} e^{i\boldsymbol{\omega} \cdot \mathbf{x}}, \quad (6)$$

where  $D$  is the domain over which the input features are encoded. Since understanding the behaviour of the Fourier coefficients given in Equation (6) is the central objective, to this end, we perform both analytical and numerical simulations in the subsequent sections.

## 3 Main Results

To understand the expressivity of the Fourier coefficients under amplitude embedding, we conduct analytic derivations of their first and second moments with respect to the parameter space under non-negative encodings. Hence, we show that, similar to angle embedding, amplitude embedding also exhibits exponential decay of expressivity with frequency, but with subtle differences in the precise scaling behaviour. Consider a VQC model defined as in Equation (1) with  $L$  layers,  $n$  qubits, with non-negative amplitude embedding featuring  $d = 2^n$  input dimensions and  $x_i \in [0, 2\pi]$ , and the parametrized unitary defined in Equation (5). Assuming the image of the parameter space  $\text{Im}(\mathcal{W})$ ,  $\mathcal{W} : \Theta \rightarrow \mathcal{U}$  within the unitary group  $\mathcal{U} \subset U(d)$  is a 2-design,  $O$  is traceless, and that the circuit undergoes a general noise channel conditioned on unitary Kraus operators,  $K_i \in U(d)$ ,  $\mathcal{E}(\rho) = \sum_k p_k E_k \rho E_k^\dagger$ ,

where  $\sum_k p_k = 1$ ,  $\sum_k E_k E_k^\dagger = \mathbb{1}$ , then the corresponding Fourier coefficients as defined in Equation (6) will have identically zero mean and exponentially decaying variance with respect to the frequency magnitudes:

$$\mathbb{E}_{\boldsymbol{\theta}}(c_{\boldsymbol{\omega}}(\boldsymbol{\theta})) = 0, \quad \text{Var}_{\boldsymbol{\theta}}(c_{\boldsymbol{\omega}}(\boldsymbol{\theta})) = \frac{d}{d^2 - 1} \left[ \frac{\text{I}(\boldsymbol{\omega})}{(2\pi)^{2d}} - \frac{1}{d} \delta_{\boldsymbol{\omega}}^0 \right] \left( \sum_k p_k^2 \right)^Q, \quad (7)$$

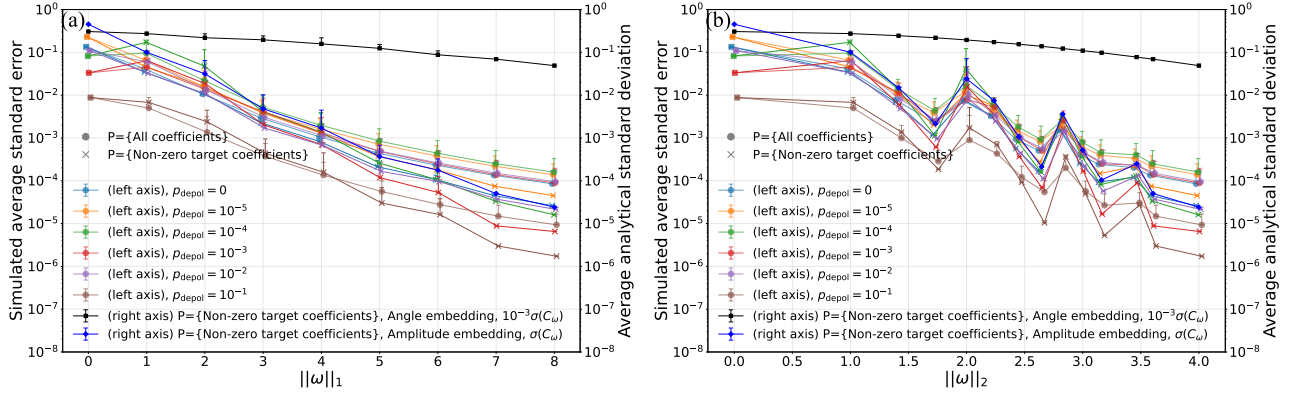


Fig. 2: (a) Left axis: empirical mean of the standard errors of the Fourier coefficients versus the  $L_1$  norm of their frequency,  $\|\omega\|_1$ . Each standard error is computed as  $\sigma(c_\omega) = \sqrt{\mathbb{E}_{t \in T}[|c_\omega|^2] - |\mathbb{E}_{t \in T}[c_\omega]|^2}$ , where  $T$  denotes the set of training iterations. After computing  $\sigma(c_\omega)$  for each frequency individually, the results are grouped by the norm  $\|\omega\|_1$  and by membership in a set  $P$ , yielding the conditional mean  $\mathbb{E}_{\omega: \|\omega\|_1 = \ell, \omega \in P}[\sigma(c_\omega)]$ . Two choices of  $P$  are considered:  $P = \{\omega : c_\omega \neq 0\}$  (non-zero target coefficients only, shown as crosses) and  $P = \Omega$  (all coefficients, shown as dots). For visual clarity and to avoid infinitely long error bars on the log scale, only the upper half of each error bar is displayed. Results are shown for depolarizing noise strengths ranging from  $p = 0$  to  $p = 10^{-1}$ , distinguished by colour. Right axis: analytically predicted standard deviations, averaged over the same frequency groups, computed from the exact amplitude-embedding variance derived in Equation (7) with  $p = 0$  and from the angle-embedding variance upper bound (scaled by  $1/1000$  to fit the plot) given in Equation (4). (b) Same as (a), but with the  $L_2$  norm  $\|\omega\|_2$  on the horizontal axis.

where  $I(\omega) = \sum_{i,j=1}^d \left| \int d^{(d)} V \frac{e^{-i\omega \cdot x} x_i x_j}{\|\mathbf{x}\|_2^2} \right|^2$ . The factor  $\left( \sum_k p_k^2 \right)^Q \leq 1$  arises from the noise and exponentially suppresses the variance as the number of instances the noise channel is applied,  $Q$ , grows. In Figure 2, we also present a simulation with depolarizing noise that compares the model's performance and sampled Fourier coefficients against the analytical results derived above. The simulation curves show good agreement with the analytical predictions, in both overall magnitude (for noiseless simulation) and the detailed exponentially decaying trend. In particular, for Figure 2a, the curves from simulation almost follow a straight line, reproducing the noiseless analytical outcome of  $O(\exp(-\|\omega\|_1))$  in the blue curve, despite differences in magnitudes coming from the noise. This alignment with the prediction also demonstrates the credibility and versatility of the premises made: Even if the analytical result is averaged over the full parameter space, whereas the simulation only averages over the parameters visited along the training trajectory, the agreement in the plot implies the impact on the variance is very limited. We also observe that it decays faster than the case of angle-embedding, as indicated by the black curves, given the same frequency norms. However, as the frequency increases, the decay of the angle-embedding standard deviation upper bound starts to accelerate, indicating that the angle-embedded model would progressively become as expressive at high frequencies. This provides a deeper understanding of how different encoding strategies have expressivity, hence trainability emphasis on different frequencies, yet the decays at an exponential rate still govern at a high level. As expected, the higher the strength of the depolarizing noise, the more the standard error of the sampled Fourier coefficients is suppressed. This alignment with the prediction speaks to its credibility and versatility, given that the analytical result is derived by averaging over the full parameter space, whereas the simulation averages only over the parameters visited along the training trajectory.

Beyond the space of this extended abstract, our work [6] also demonstrates other few minor outcomes, including how differences in the domain of the amplitudes have a huge impact on the zeroth Fourier coefficient, and how an amplitude-embedded model can train a function with non-integer frequencies while angle-embedded ones cannot.

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