

Classical shadows over symmetric spaces

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A more technical version of this work can be found at <https://arxiv.org/pdf/2605.05518>

Introduction

Determining properties of unknown quantum states is both a fundamental task in quantum information theory and of rapidly increasing practical relevance. In recent years, as the surging size of experimentally accessible quantum systems has forced the development of techniques beyond full state tomography, the theory of *classical shadows* [1] has emerged as a leading framework within which to perform such learning. In a classical shadows protocol one measures samples of an unknown target state ρ in randomised bases dictated by a protocol-dependent ensemble $\mathcal{E}$ of unitaries, and uses the resulting data to reconstruct “shadows” of ρ , which are in turn used as proxies against which observables can be estimated. For a given target set of observables, the (sample-)efficiency with which this can be done depends strongly on the selected ensemble of unitaries, with an explosion of research into the various benefits of different ensembles having quickly appeared [1–12].

Although the properties of classical shadow protocols are now quite well understood when the ensemble $\mathcal{E}$ of unitaries from which one is sampling forms a compact group [2], there has been little exploration of the behaviour of classical shadow protocols beyond that assumption [10]. In this work we study the classical shadow protocols induced by sampling uniformly randomly from the seven infinite families of *compact symmetric spaces of type I* [13]. Such symmetric spaces can be thought of as a quotient G/K , with K a (generically non-normal) subgroup of a compact Lie group G satisfying various conditions. Symmetric spaces have long been recognized to play a fundamental role in the analysis of symmetry in quantum mechanics [14, 15]; more recently, they have found technical applications in quantum compilation [16–20]. The classical shadow protocol induced by sampling uniformly from G/K may be interpreted as the protocol induced by sampling from a certain non-uniform distribution on G . Despite the lack of a group structure, some calculation gives us a unifying theory of such protocols. We find that the classical shadow *measurement channel* associated to G/K is a convex combination of the channel associated to G , a dephasing channel, and (when G is the symplectic group) a further subleading term. Essentially, therefore, the classical shadow protocols on G/K resemble those on G , but modified to prefer a certain basis. Interestingly, some of these protocols allow for a non-vanishing improvement in sample-complexity over the group-based ensembles for the estimation of observables sampled from distributions centred on the preferred basis, and may therefore find experimental applications.

Preliminaries

We begin by briefly recalling the key points of classical shadows; more details may be found in Refs. [1, 2]. A classical shadow protocol $(\mathcal{E}, \mathcal{W})$ is defined by two choices: an ensemble $\mathcal{E}$ of unitaries acting on the Hilbert space $\mathcal{H} \cong \mathbb{C}^d$ of the unknown state ρ , and a measurement basis $\mathcal{W} = \{|w\rangle\}_w$ of $\mathcal{H}$. Having made these choices, one takes copies of one’s unknown state, acts with a randomly

drawn $U \sim \mathcal{E}$, and measures in the basis $\mathcal{W}$. Upon obtaining the outcome (say) $|w\rangle$, one classically stores the state $U^\dagger |w\rangle\langle w| U$. The result of all this is to effect the *measurement channel*

$$\mathcal{M}_{\mathcal{E},\mathcal{W}} = \mathbb{E}_{U \sim \mathcal{E}} \text{Ad}_{U^\dagger} \circ \mathcal{A}_{\mathcal{W}} \circ \text{Ad}_U, \quad (1)$$

where $\mathcal{A}_{\mathcal{W}}(-) = \sum_w \langle w|(-)|w\rangle |w\rangle\langle w|$ is a dephasing channel with respect to $\mathcal{W}$ and $\text{Ad}_U(-) = U(-)U^\dagger$ the adjoint action of U . The (pseudo-)inverse of $\mathcal{M}_{\mathcal{E},\mathcal{W}}$ on the states generated in this way from a collection of N copies of ρ forms a set $\{\mathcal{M}_{\mathcal{E},\mathcal{W}}^{-1}(U_i^\dagger |w_i\rangle\langle w_i| U_i)\}_{i=1}^N$ of *classical shadows* of ρ which one uses as proxies $\{\hat{\rho}_i\}_{i=1}^N$ against which to evaluate expectation values of observables O ,

$$\text{tr}[\rho O] \approx \frac{1}{N} \sum_{i=1}^N \text{tr}[\hat{\rho}_i O]. \quad (2)$$

Two natural questions arise: (i) are the estimators of Eq. (2) unbiased, and (ii) how many shadows does one need to (with a certain probability) obtain estimates to a given target precision? The answers to these questions depend on the set of observables which one is interested in measuring, and that (conditioned on the selection of observables) different choices of $\mathcal{E}$ and $\mathcal{W}$ can lead to protocols of varying favorability. As previously stated, the point of this work is to investigate the effects of taking $\mathcal{E}$ to be the uniform measure on a classical compact symmetric space.

Definition 1. A classical compact symmetric space (of type I) is a quotient G/K , where G is the unitary, orthogonal or (unitary) symplectic group, and K is the fixed point set of an involution $\sigma : G \rightarrow G$.

All of the possible spaces of this form were classified by Cartan [21]; there are seven infinite families. Importantly for our purposes, the uniform measure on a compact symmetric space of the above form has a simple algorithmic description [22, 23]; a random $t \sim G/K$ is given by $\sigma(g)^{-1}g$, where g is drawn uniformly randomly from the ambient group G . For example, a random element of AI can be produced by generating a random $U \in \mathbb{U}(d)$ and then “returning” $\sigma_{\text{AI}}(U)^\dagger U = (U^*)^\dagger U = U^\top U$. What we will in effect be doing, then, is studying the classical shadow protocols induced by certain non-invariant measures on the unitary, orthogonal, and unitary symplectic groups.

Results

The application of tools from representation theory has allowed the classical shadow protocols resulting from sampling uniformly from a compact group G to become quite well understood [2]. For example, an elementary but powerful observation is that in this setting the measurement channel Eq. (1) is manifestly G -equivariant: for any $g \in G$, $\mathcal{M}_{G,\mathcal{W}} \circ \text{Ad}_g = \text{Ad}_g \circ \mathcal{M}_{G,\mathcal{W}}$. It immediately follows from Schur’s lemma [24] that one has the decomposition

$$\mathcal{M}_{G,\mathcal{W}} \cong \bigoplus_{\lambda} \mathbb{1}_{d_{\lambda}} \otimes s_{\lambda}, \quad (3)$$

where the direct sum is over the irreps of G in $\text{End } \mathcal{H}$. Here d_{λ} is the dimension of the irrep λ (which appears in the decomposition with some multiplicity m_{λ}) and the $s_{\lambda} \in \text{End } (\mathbb{C}^{m_{\lambda}})$ are to-be-determined operators.

In contrast to this group-structure-induced behaviour, we show that when we take $\mathcal{E}$ to be the uniform measure on a symmetric space it is *not* the case that the resulting measurement channel commutes with the action of an arbitrary coset representative in G/K ; a somewhat weaker result,

however, is true. To see it, let us say a group H is $\mathcal{W}$ -normalizing if, for all $h \in H$ and $|w\rangle \in \mathcal{W}$, there exists a $|w'\rangle \in \mathcal{W}$ such that $h|w\rangle\langle w|h^\dagger = |w'\rangle\langle w'|$. We will use $N_{\mathcal{W}}$ to denote the group of all $\mathcal{W}$ -normalizing elements of G . We find:

Lemma 1. *The classical shadows measurement channel $\mathcal{M}_{G/K, \mathcal{W}}$ resulting from sampling uniformly from G/K and measuring in a basis $\mathcal{W}$ commutes with the adjoint action of any group $H \subseteq K \cap N_{\mathcal{W}}$, so that*

$$\mathcal{M}_{G/K, \mathcal{W}} = \bigoplus_{\lambda} \mathbb{1}_{d_{\lambda}} \otimes s_{\lambda}; \quad (4)$$

where the direct sum is over the irreps of H in $\text{End } \mathcal{H}$.

As we shall soon see, we further find that for all our examples the decomposition of Eq. (4) is *multiplicity-free*; that is, $s_{\lambda} \in \mathbb{R} \forall \lambda$. This allows one to trivially invert the measurement channels, avoiding the *a priori* need for extensive classical post processing [2]. These calculations are facilitated by the simple observation that the measurement channel Eq. (1) can be re-expressed as

$$\mathcal{M}_{\mathcal{E}, \mathcal{W}}(\rho) = \text{tr}_1 \left[(\rho \otimes \mathbb{1}) \mathcal{T}_{\mathcal{E}}^{(2)} \left(\sum_{w \in \mathcal{W}} |w\rangle\langle w|^{\otimes 2} \right) \right], \quad (5)$$

where $\mathcal{T}_{\mathcal{E}}^{(k)} : \mathcal{L}^{\otimes k} \rightarrow \mathcal{L}^{\otimes k}$, $\mathcal{T}^{(k)}(A) = \mathbb{E}_{U \sim \mathcal{E}} U^{\otimes k} A (U^\dagger)^{\otimes k}$. Our immediate task, then, is an exercise in evaluating second-order twirls over symmetric spaces. It follows from earlier discussion that one can express these integrals as *fourth-order* integrals over the ambient group, thus reducing the task to standard Weingarten calculus [25] on the unitary, orthogonal and symplectic groups [26]. Alternately, it was shown by Matsumoto [23] that one can integrate over the symmetric spaces directly, without explicitly passing to the parent group; we explore both of these options. We find:

Theorem 1. *Sampling uniformly from G/K and measuring in the basis $\mathcal{W}$ with respect to which the involutions take their standard forms yields*

$$\begin{aligned} \mathcal{M}_{G/K, \mathcal{W}}(\rho) &= (1 - \alpha_{G/K}) \mathcal{M}_{G, \mathcal{W}}(\rho) + \beta_{G/K} \mathcal{A}_{\mathcal{W}}(\rho) \\ &\quad + (\alpha_{G/K} - \beta_{G/K}) \left(J \mathcal{A}_{\mathcal{W}}(\rho) J^\dagger - \mathcal{A}_{\mathcal{W}}(\rho J) J \right), \end{aligned} \quad (6)$$

where $\mathcal{A}_{\mathcal{W}}$ is a dephasing channel onto $\mathcal{W}$ and J is the canonical symplectic form. For AI, AII, CI, and DIII, $\alpha_{G/K} \in \mathcal{O}(d^{-2})$; for AIII, BDI and CII, $0 \leq \alpha_{G/K} \leq 1$ is tunable. Finally, $\alpha_{G/K} = \beta_{G/K}$ for $G \neq \mathbb{SP}$, and $|\alpha_{G/K} - \beta_{G/K}| \in \mathcal{O}(1/d)$ for $G = \mathbb{SP}$.

Firstly, despite the lack of a group structure, the shadow channel over any of these symmetric spaces may be written in terms of only two *a priori* unknown parameters (and indeed for $G \neq \mathbb{SP}$ we have $\alpha_{G/K} = \beta_{G/K}$, so that there is only one unknown parameter). This observation obviates the need for performing the full calculation of Eq. (5) explicitly, as one can fix the unknown coefficients by looking at some specific matrix entries; this easier route is provided by Theorem 1.

Secondly, from their respective coefficients $\alpha_{G/K}$ being all of order d^{-2} , it follows immediately from Theorem 1 that the classical shadow protocols induced by AI, AII, CI, and DIII are unlikely to be of practical interest, except insofar as they inherit the benefits of the known protocols over the unitary [1], orthogonal [6], and symplectic [9] groups which they approximately reproduce. Separately, and though not entirely immediate, it follows from the twin facts that the symplectic group induces unitary state k -designs for all k [9] and that $\alpha_{\text{CII}} - \beta_{\text{CII}} \in \mathcal{O}(1/d)$ that the AIII and CII protocols coincide to leading order. For observables with non-negligible concentration on the diagonal subspace (with respect to the basis of the dephasing channel) we find that the AIII and BDI protocols offer modest improvements over the unitary and orthogonal protocols respectively.

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