

# A journey through Flatland: What does the antiflatness of a spectrum teach us?

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**Introduction.** A basic problem in quantum learning theory is to determine which properties of an unknown quantum system can be inferred efficiently from experimental access to that system. In a state-learning formulation, one is given copies of an unknown state  $\rho \in \mathcal{D}(\mathcal{H})$ , and the task is to estimate selected properties of  $\rho$  to prescribed accuracy and confidence. In a Hamiltonian-learning formulation, the unknown object is an operator  $H = \sum_{a=1}^m \lambda_a E_a$ , where the  $E_a$  are local interaction terms and the  $\lambda_a$  are real coefficients. The task is to estimate either the coefficients  $\lambda_a$ , the interaction structure, or selected functions of  $H$ , using access to Gibbs states, real-time evolution, or other experimental primitives. Recent results show that the objective is usually not full tomography of an exponentially large quantum object, but rather the efficient inference of structured information from a restricted class of states, observables, or dynamics [1, 2, 3, 4, 5]. The relevant notion of efficiency is therefore not only computational time, but also the number of samples, measurements, or uses of a quantum process required to solve the task. This is the sample complexity of the learning problem. Full quantum state tomography is exponentially expensive in the number of qubits in the absence of additional assumptions. By contrast, structured learning tasks can be efficient. Examples include learning local Hamiltonians from high-temperature Gibbs states [3], learning Hamiltonian structure from real-time evolution [4], and predicting many observables of a quantum state using classical shadows [2]. Given that learning theory is concerned with identifying structural parameters that control the complexity of inference, one may ask whether the internal structure of the entanglement spectrum can also define parameters relevant to the learnability of quantum states. Quantum resource theory provides the complementary language: it specifies which states and operations are free, which objects are resourceful, and which monotones constrain state conversion. General resource theories formalize this by dividing states into free and resource states and studying what transformations are possible under restricted operations. Quantum resources are not only physical properties, but also learning-theoretic parameters controlling the richness of hypothesis classes.

**Contribution.** In the presented contribution, we study a structural property of bipartite quantum states that is not captured by standard entanglement measures. Given a Hilbert space equipped with a tensor product structure such that  $\mathcal{H} \simeq \mathcal{H}_A \otimes \mathcal{H}_B$  and a pure state  $|\psi\rangle_{AB} \in \mathcal{H}$ , we consider the reduced density matrix  $\rho_A = \text{Tr}_B |\psi\rangle\langle\psi|$  and its spectrum  $\lambda(\rho_A) = (\lambda_1, \dots, \lambda_r)$ . This spectrum, equivalently the spectrum of the modular Hamiltonian  $K_A = -\log \rho_A$ , is called the entanglement spectrum (ES). The ES has proved to be a powerful diagnostic tool; it allows one to identify topological order through edge-state correspondence [6, 7, 8], extract scaling dimensions in conformal field theories [9, 10], and distinguish ergodic thermalization from many-body localization [11, 12]. We study the properties of the ES through the lens of antiflatness [13]. The set of *flat*<sub>A</sub> states is defined as the set of those states whose reduced density matrix is maximally uniform on its support, i.e., proportional to a projector

$$\text{flat}_A := \left\{ \rho \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B) \left| \rho_A^2 = \frac{\rho_A}{\text{rank}(\rho_A)} \right. \right\}. \quad (1)$$

Equivalently, a state belongs to *flat*<sub>A</sub> if and only if it can be written as  $\rho_A = \frac{\Pi_A}{\text{rank}(\Pi_A)}$ , where  $\Pi_A$  is the projector onto the support of  $\rho_A$ . This condition implies that all Rényi entropies of  $\rho_A$  coincide, pro-

viding an information-theoretic characterization of antifatness. States that do not satisfy this condition exhibit non-uniform eigenvalue distributions in their reduced density matrix and are therefore referred to as *antiflat<sub>A</sub> states*. Physical examples of flat<sub>A</sub> states are provided by stabilizer states [14], or systems with topological order, where the ES may display exact or approximate degeneracies associated with long-range entanglement and edge-state correspondence [6, 8, 15, 16, 17]. Deviations from a flat<sub>A</sub> spectrum quantify the emergence of nontrivial spectral organization that can be used to diagnose topological order [18, 19]. Our fundamental motivation is to use antifatness as a bridge to explore the interplay between entanglement and magic [20, 21, 22]. For a bipartition of a pure state, a non-trivial ES is a direct signature of entanglement. At the same time, stabilizer states are characterized by a flat ES (yielding zero antifatness) [14]. Deforming this spectrum strictly requires the presence of both entanglement and magic, even though their mere coexistence is not always sufficient [23]. For example, it is possible to introduce magic by local transformations without changing the ES, and so without changing the antifatness.

Antifatness thus provides a natural framework to identify spectral features that remain invisible to standard entanglement measures but are sensitive to the computational complexity of the underlying quantum state. Quantities such as the von Neumann entropy, the purity, or Rényi entropies measure average spectral properties, but they do not determine how the spectral weights are internally distributed. These correspond to the first moment of the ES and therefore provide a way to measure the average amount of quantum correlations between subsystems [24]. Distinct spectra may share the same entropy or purity while displaying different distributions of spectral weights. This provides a complementary viewpoint on quantum state learning: rather than asking only how much entanglement or magic a state contains, we ask how these resources are distributed across the spectrum of the reduced density matrix.

**Main results.** We introduce an ordering of states based on the spread of Rényi entropies, leading to an antiflat majorization relation that is sensitive to spectral inhomogeneity rather than only to purity. This motivates a class of Flatness-Preserving Operations and gives necessary conditions for state convertibility that are invisible to standard majorization. We further show that several antifatness quantifiers, including the Capacity of Entanglement, the Linear Rényi Spread, and logarithmic antifatness, can be unified using escort distributions and information-geometric tools. The escort distributions deform a spectrum by emphasizing different eigenvalue scales. For an escort parameter larger than one, the largest eigenvalues are amplified; for a parameter smaller than one, the smaller eigenvalues become more visible. This gives a “zoom lens” on the entanglement spectrum. The escort flow can be seen as a modular-temperature deformation of the reduced density matrix. Then we reformulate antifatness using Bregman divergences. In this language, a state is compared not with an arbitrary external reference, but with its own escort-deformed versions. This makes the Capacity of Entanglement an intrinsic spectral susceptibility: it measures how distinguishable a spectrum becomes from itself under infinitesimal escort deformation. The Capacity of Entanglement can be expressed as a second derivative of the Kullback–Leibler divergence along the escort trajectory, thereby linking it to Quantum Fisher Information. This identifies antifatness as a geometric response function of the entanglement spectrum. Peaks or variations in this susceptibility can indicate reorganizations of the spectrum even when ordinary entropy does not change or changes smoothly. In standard quantum metrology, QFI measures sensitivity to an externally encoded parameter, such as a phase, a magnetic field, or a Hamiltonian coefficient. In our setting, the parameter  $\alpha$  is not an externally applied laboratory parameter, but an intrinsic spectral deformation parameter. The equality  $C_E = F_{\alpha=1}$  shows that the Capacity of Entanglement is the Fisher information associated with the escort deformation of the entanglement spectrum. This suggests that antifatness may play the role of a resource-sensitive learnability parameter for families of quantum states organized by their entanglement spectra. Finally, we analyze the typical behavior of entanglement-spectrum fluctuations under Haar, Bures–Hall, and  $t$ -doped Clifford random-state ensembles. We find that these spectral fluctuations vanish rapidly in the large-dimensional bipartite limit, refining the standard concentration-of-measure picture according to which the reduced state of a sufficiently small subsystem is overwhelmingly close to maximally mixed [25, 26]. We derive an upper bound on the rate at which linear entanglement can be generated during dynamical evolution, expressed

in terms of the Linear Rényi Spread. Our work provides a resource-theoretic vocabulary for asking when quantum states are easy or hard to learn because of the spectral organization of their entanglement and magic resources.

**Relevance for QTML.** This work connects to the scope of QTML through the theoretical foundations of quantum learning. By identifying a resource-theoretic structure in the entanglement spectrum, our results suggest that the learnability of quantum states may depend not only on Hilbert-space dimension, rank, or entropy, but also on the internal organization of quantum resources across the entanglement spectrum. For this reason, antiflatness provides a possible language for discussing how entanglement and magic affect learning problems, in line with recent works on quantum pseudo-random states [27] and quantum chaos [28]. It gives a way to ask whether two states with the same average entanglement may nevertheless have different learning complexity because their spectral resources are organized differently. Our results address a foundational question closely related to the conference: which quantum resources control the information-theoretic and computational difficulty of learning quantum systems?

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