Open quantum systems II

Non-Markovian open quantum systems

Graeme Pleasance

Quantum@SUN group
Department of Physics
Stellenbosch University

33rd Chris Engelbrecht Summer School
11 April 2025

NITheCS
National Institute for Theoretical and Computational Sciences

Stellenbosch University
Iyunivesithi
Universiteit

QSUN
Quantum Research Group of Stellenbosch University

Background

Stellenbosch
UNIVERSITY
IYUNIVESITHI
UNIVERSITEIT

QSUN
Quantum Research Group
of Stellenbosch University

Quantum Science & Technology
at Stellenbosch University

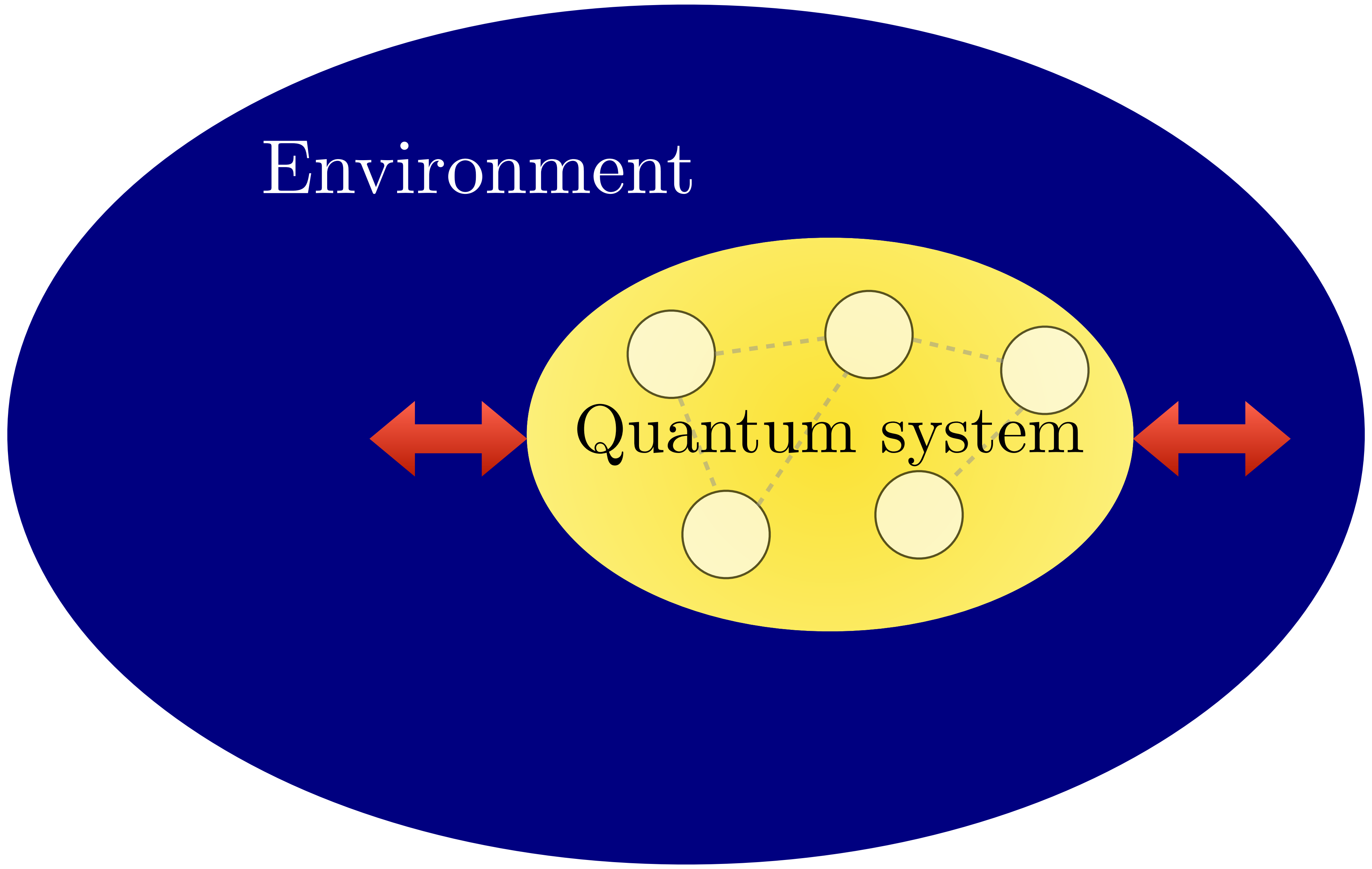
- Quantum computing
- Quantum machine learning
- Open quantum systems
- Quantum thermodynamics

Outline

- Motivations.
- Open quantum systems - Markov approximation.
- Markovianity in classical stochastic processes.
- Quantum non-Markovianity - divisibility, distinguishability.
- Example: Spontaneous emission of two-level system.
- Collision models.

Motivation

Open quantum systems - a quantum system that interacts with a surrounding environment.

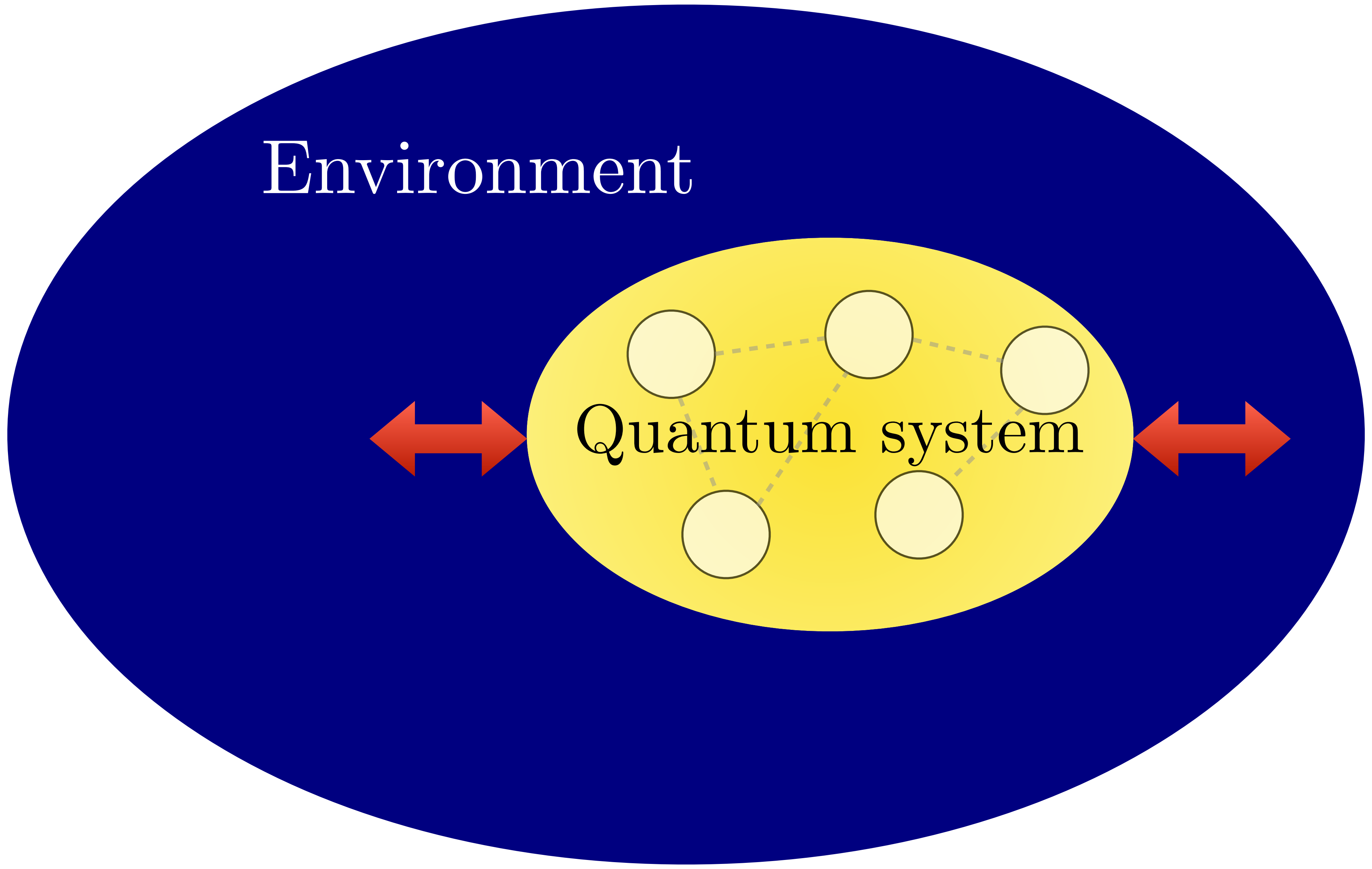


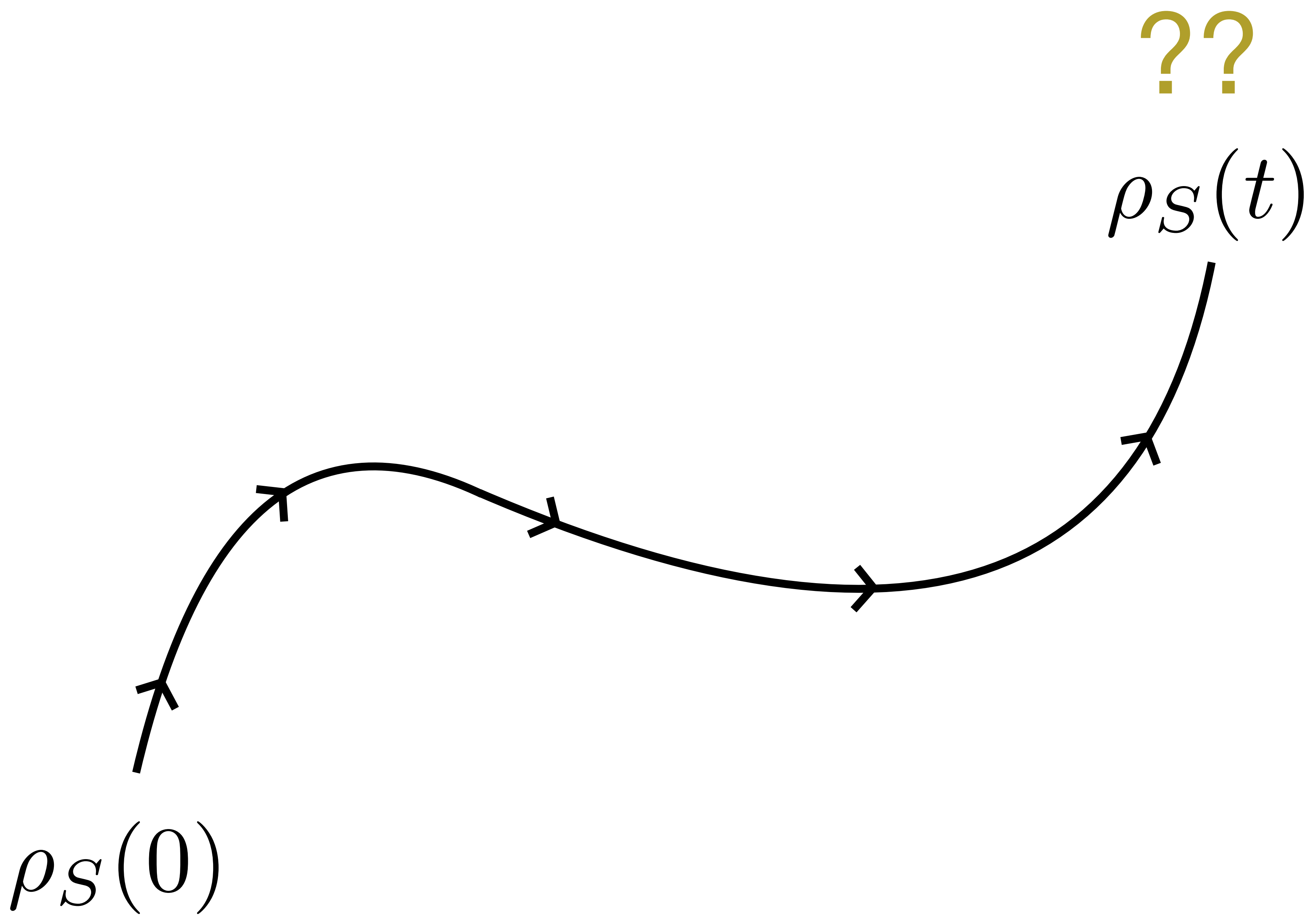
Theory of open quantum systems - framework to describe the time-dependent behavior of the system.

Motivation

Stellenbosch
UNIVERSITY
IYUNIVESITHI
UNIVERSITEIT

Open quantum systems - a quantum system that interacts with a surrounding environment.

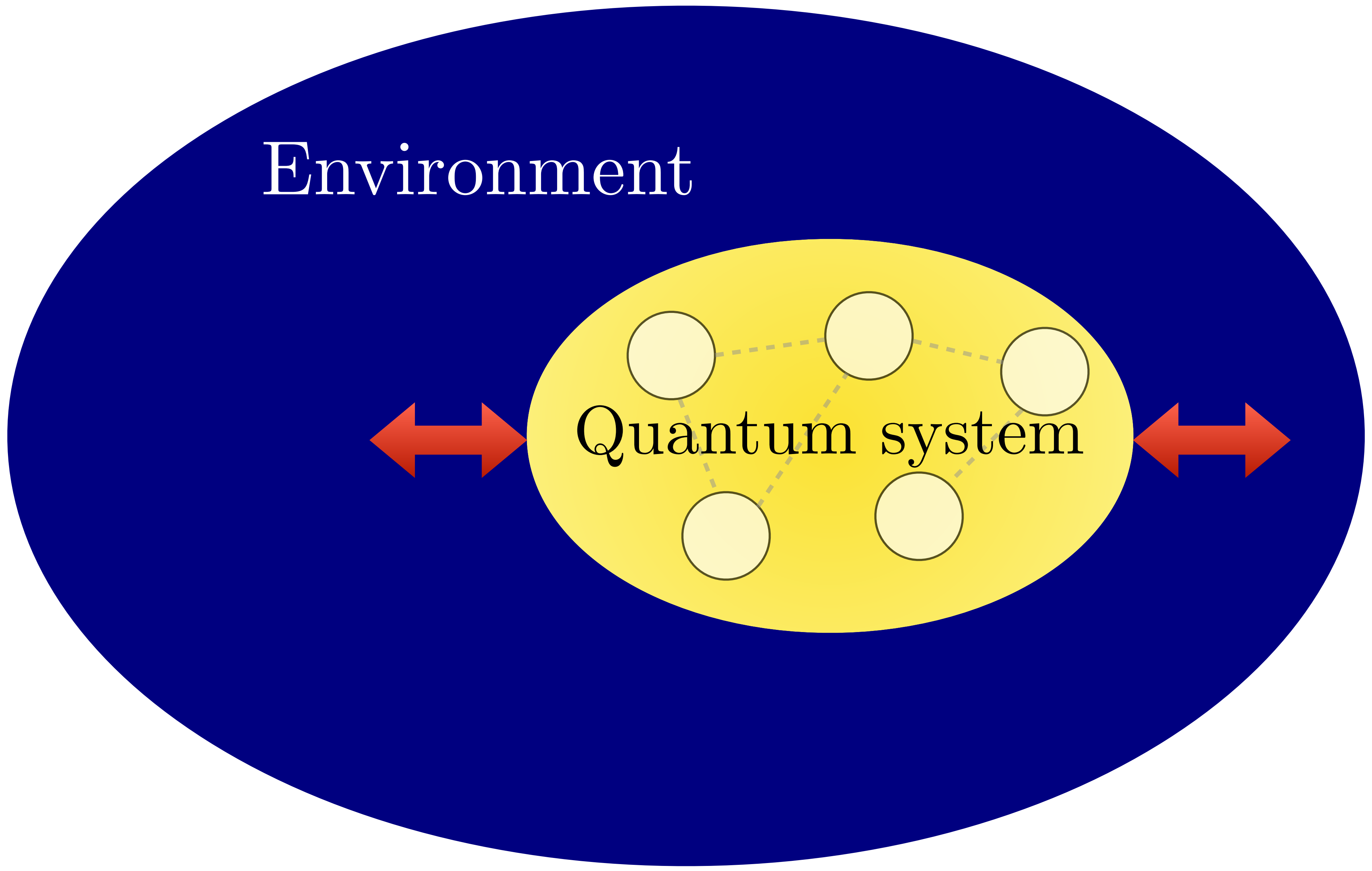


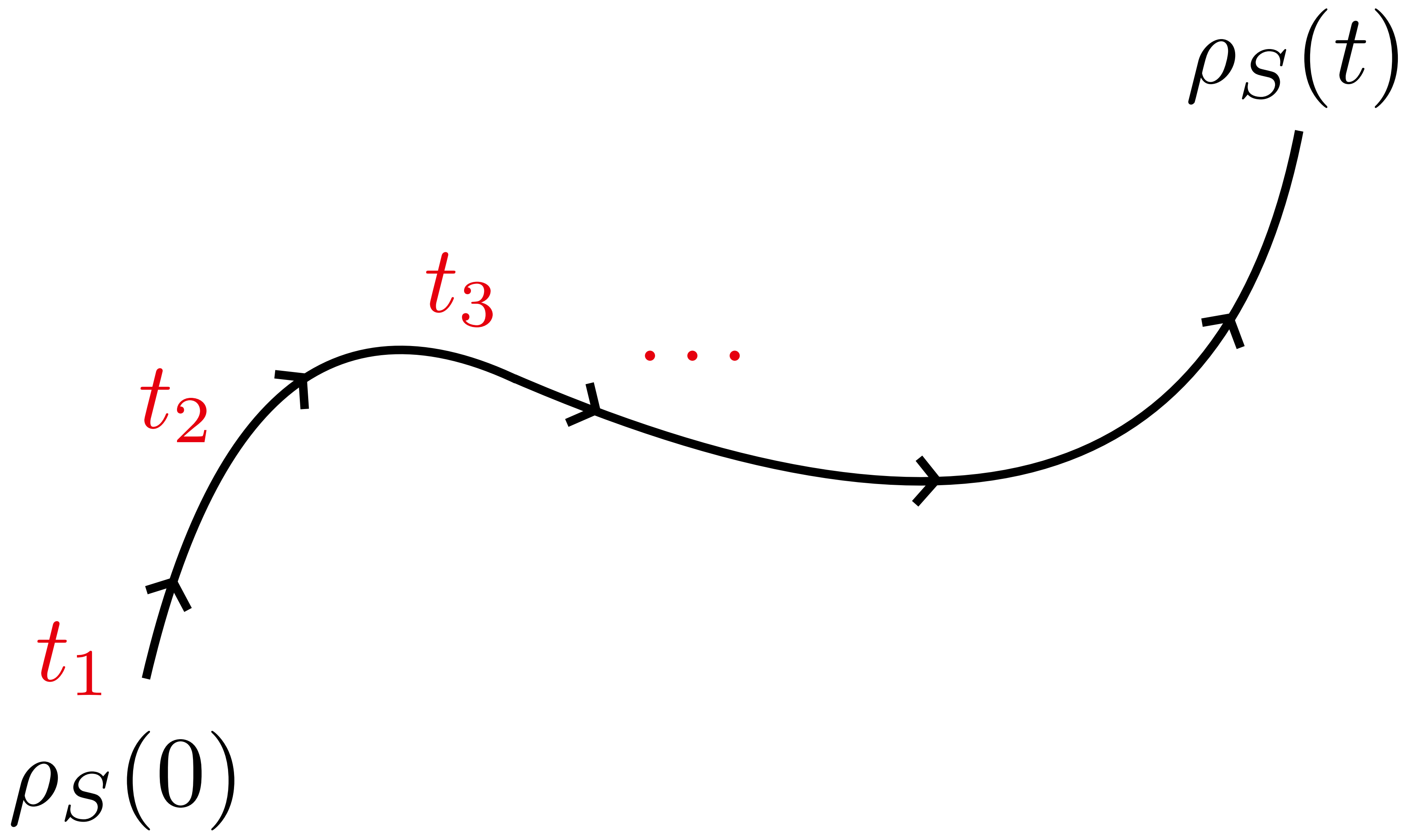


Theory of open quantum systems - framework to describe the time-dependent behavior of the system.

Motivation

Non-Markovian open quantum systems - time-dependent behavior influenced by memory effects.





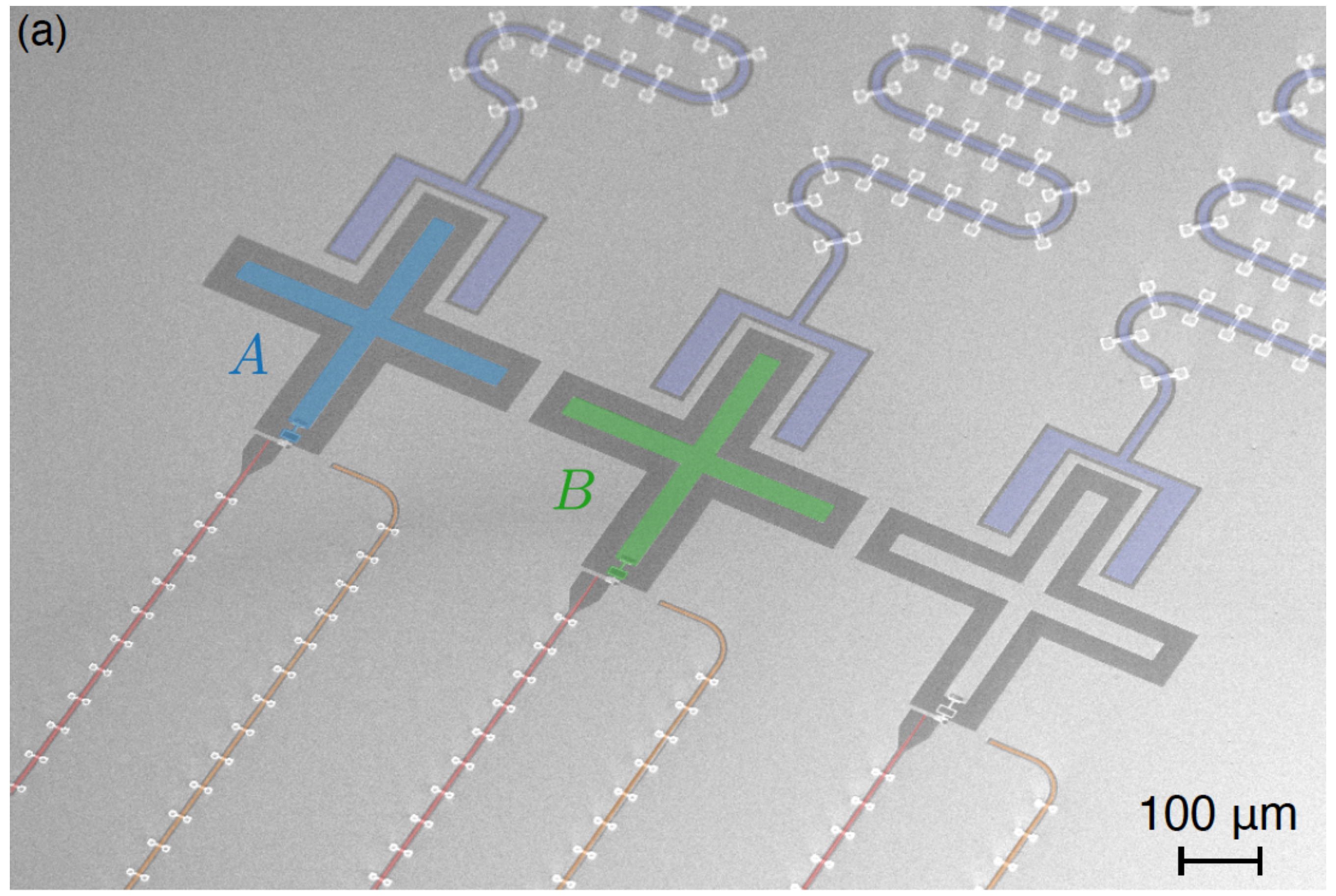
Motivation

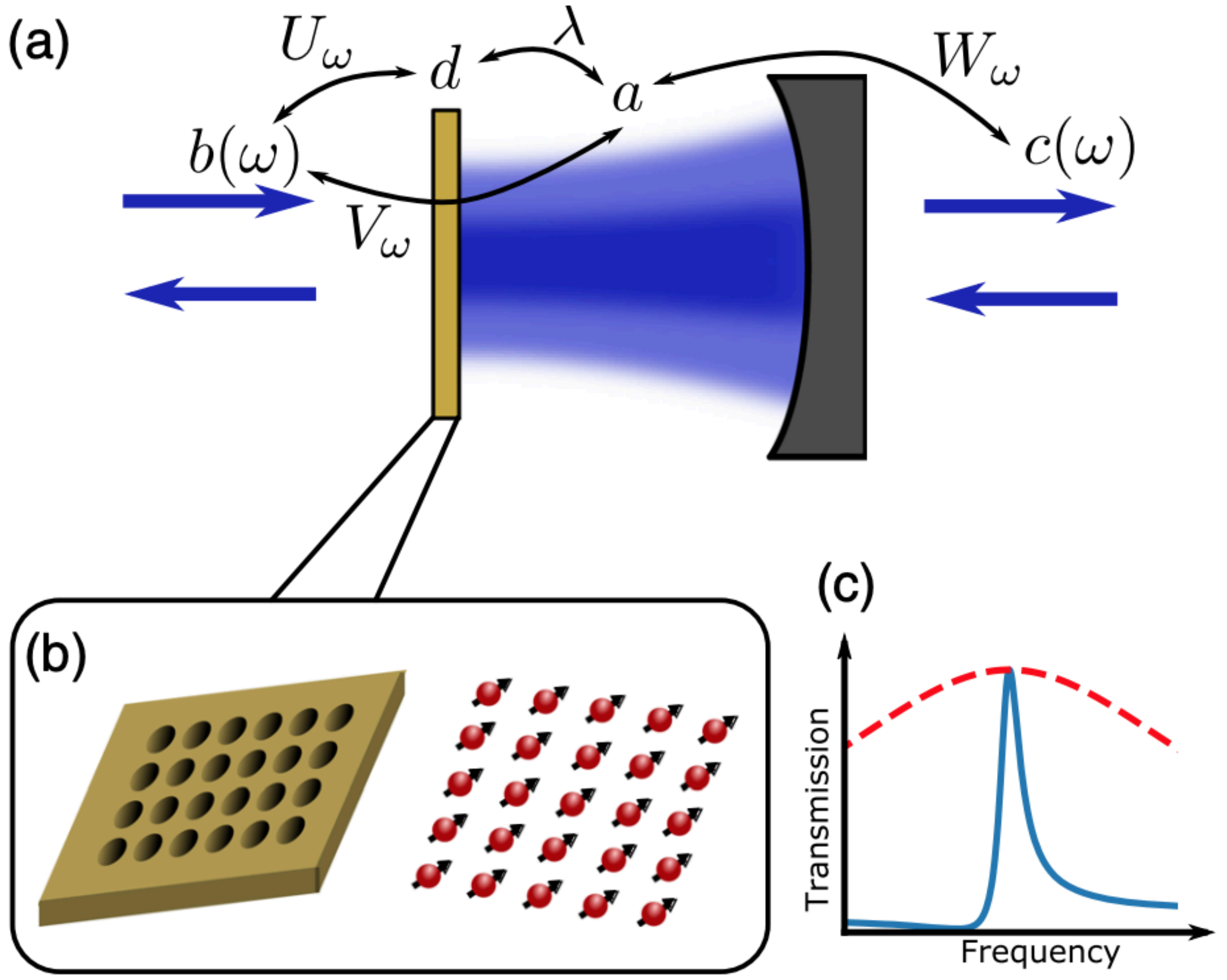
Non-Markovian open quantum systems - time-dependent behavior influenced by memory effects.

- Superconducting qubits:
  (Samach et. al., PR App 18, 064056, 2022)
  (Zhang et. al., PR App 17, 054018, 2022)

- Quantum biological systems:
  (Chin et. al., Nat Phys 9, 113, 2013)
  (Ishizaki and Fleming, Proc Nat. Acad Sci, 106, 17255, 2009)

- Cavity quantum electrodynamics:
  (Cernotik, et. al., PRL, 122, 243601, 2020)
  (Denning, Iles-Smith, Mork, PRB, 100, 214306 2019)





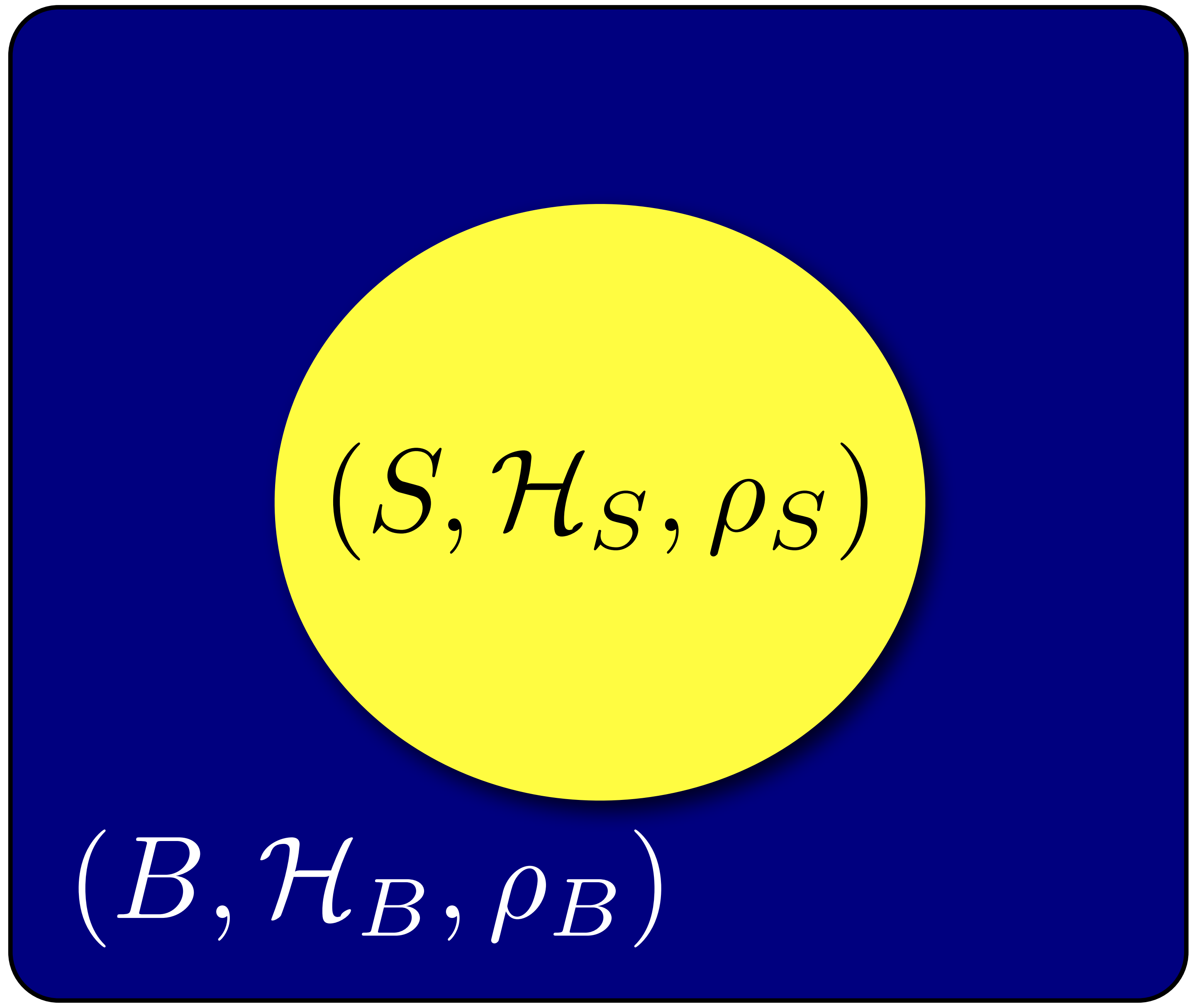
Open quantum systems

System-environment composite $\psi \in \mathcal{H}$:

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$$

Physical states ρ: $\rho \succeq 0, \quad \rho = \rho^\dagger, \quad \mathrm{Tr}\rho = 1$

$(S + B, \mathcal{H}_S \otimes \mathcal{H}_B, \rho)$



Open quantum systems

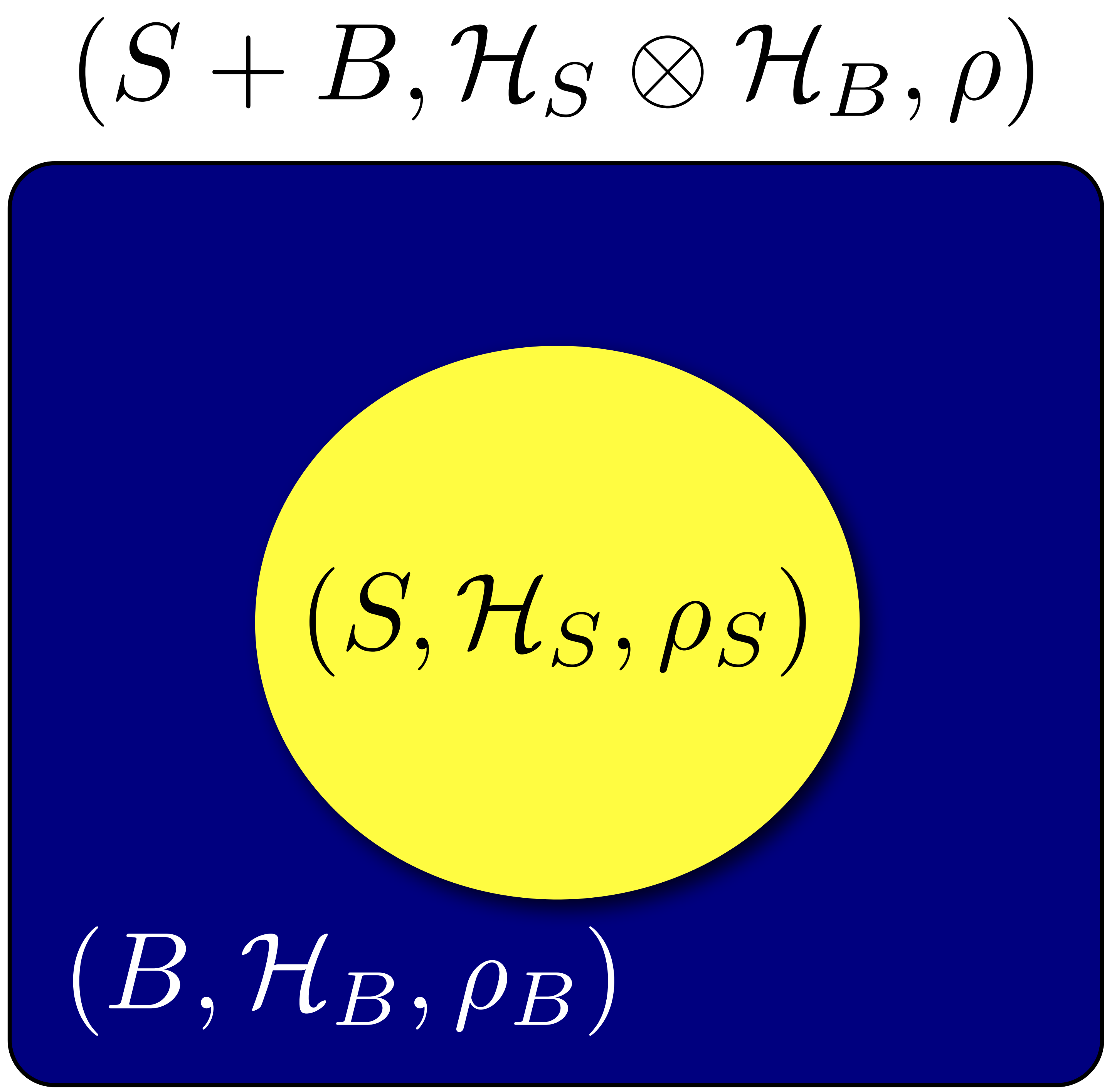
System-environment composite $\psi \in \mathcal{H}$:

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$$

Physical states ρ: $\rho \succeq 0, \quad \rho = \rho^\dagger, \quad \mathrm{Tr}\rho = 1$

Pure states: $$|\psi\rangle \iff \rho = |\psi\rangle\langle\psi|$$

Mixed states: $$\{p_j, |\psi_j\rangle\} \iff \rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$$



Open quantum systems

System-environment composite $\psi \in \mathcal{H}$:

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$$

Physical states ρ: $\rho \succeq 0, \quad \rho = \rho^\dagger, \quad \mathrm{Tr}\rho = 1$

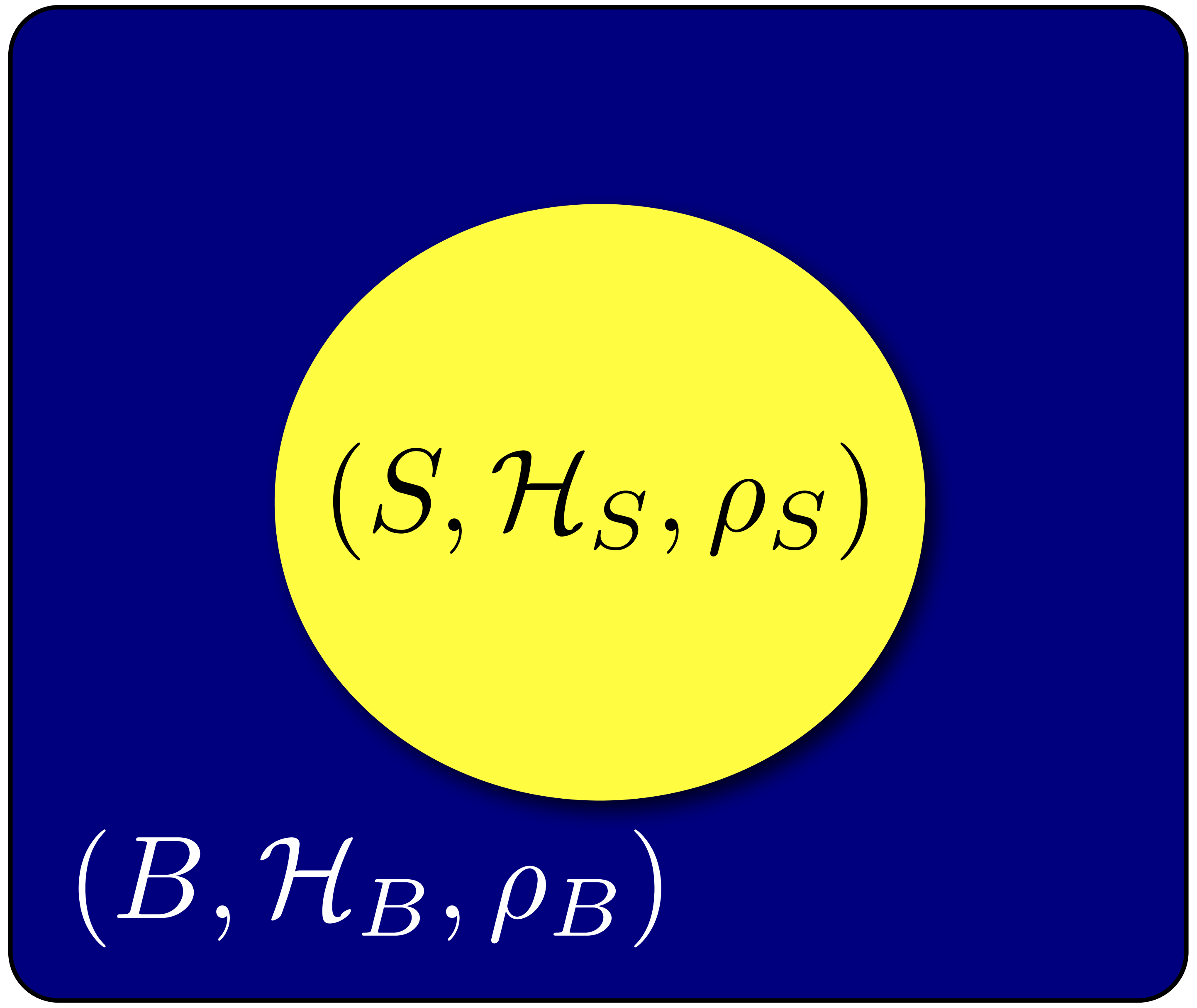
Pure states: $|\psi\rangle \Longleftrightarrow \rho = |\psi\rangle\langle\psi|$

Mixed states: $\{p_j, |\psi_j\rangle\} \Longleftrightarrow \rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$

Partial trace:

$$\rho_S = \mathrm{Tr}_E[\rho]$$

$(S + B, \mathcal{H}_S \otimes \mathcal{H}_B, \rho)$



Contains all statistical information on system S

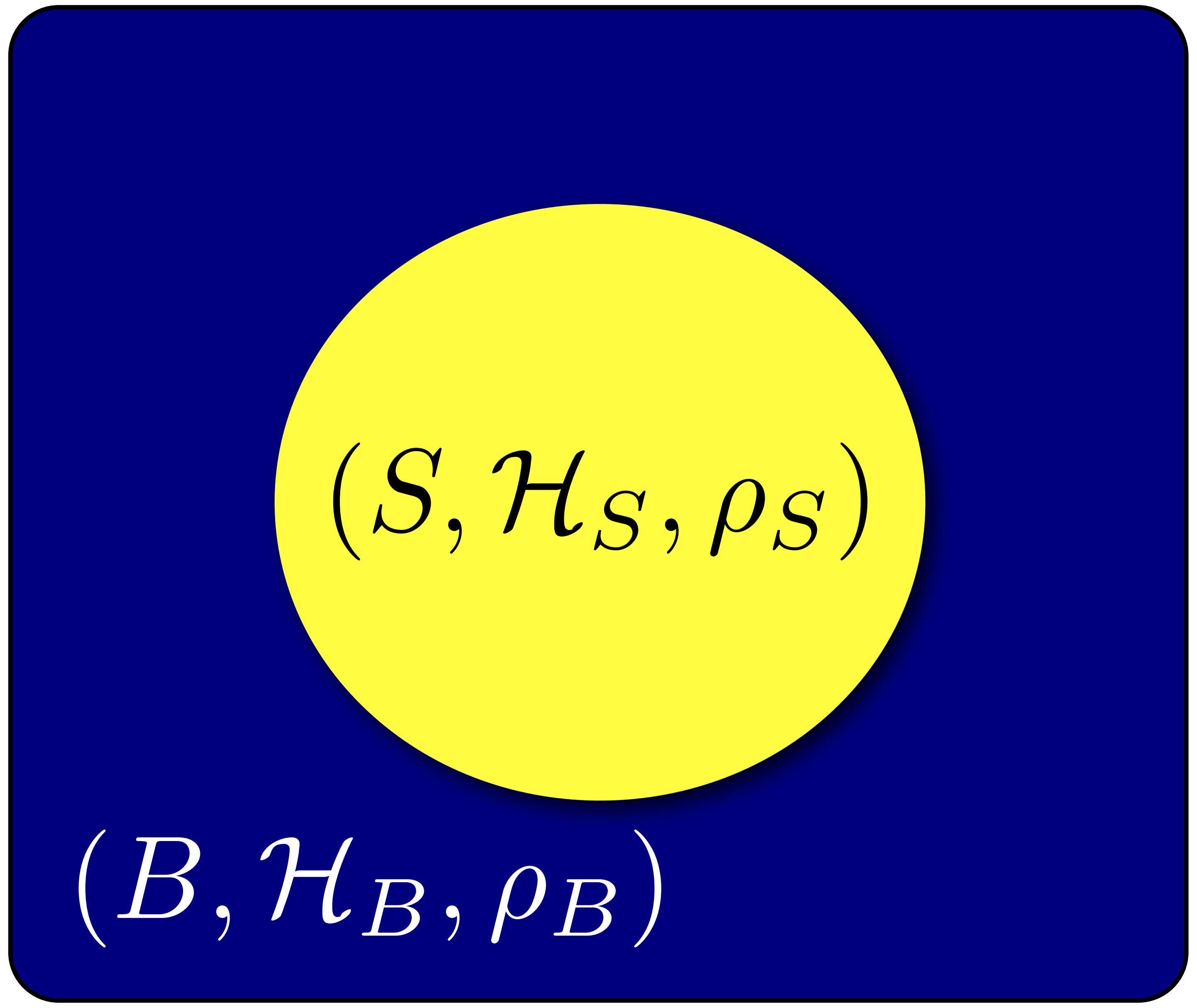
Open quantum systems

Time evolution: $|\psi_j(t)\rangle = e^{-iHt}|\psi_j(0)\rangle$

Hamiltonian:

$$H = H_S \otimes \mathbb{I}_E + \mathbb{I}_S \otimes H_E + H_I$$

$$(S + B, \mathcal{H}_S \otimes \mathcal{H}_B, \rho)$$



Open quantum systems

Stellenbosch
UNIVERSITY
IYUNIVESITHI
UNIVERSITEIT

Time evolution: $|\psi_j(t)\rangle = e^{-iHt}|\psi_j(0)\rangle$

Hamiltonian:

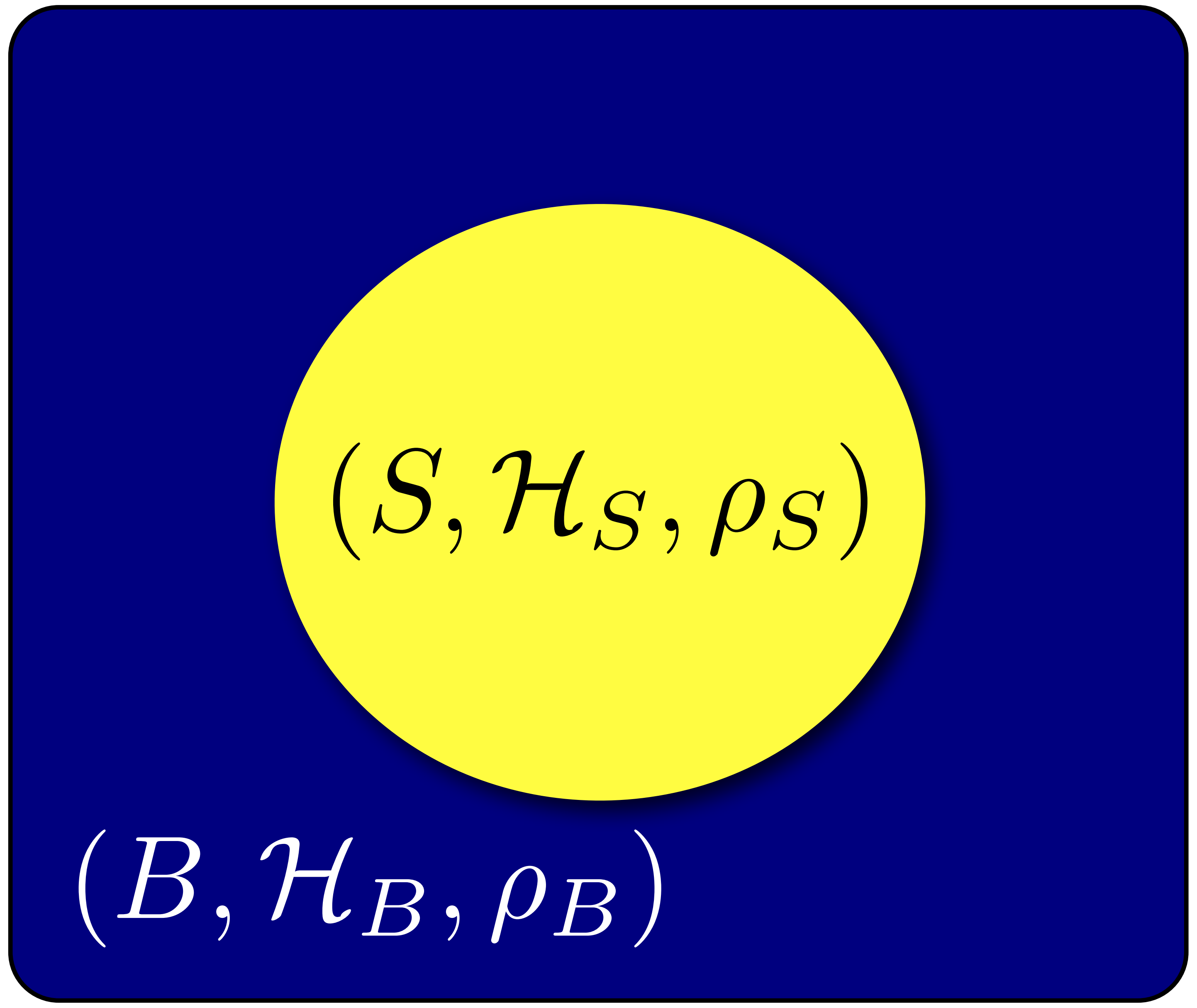
$$H = H_S \otimes \mathbb{I}_E + \mathbb{I}_S \otimes H_E + H_I$$

Interaction picture:

$$\tilde{\rho}(t) = U_I(t)\rho(0)U_I^\dagger(t) \qquad U_I(t) = \mathcal{T}e^{-i\int_0^t ds\, \tilde{H}_I(s)}$$

$$\tilde{H}_I(t) = e^{i(H_S+H_E)t} H_I e^{-i(H_S+H_E)t}$$

$(S+B, \mathcal{H}_S \otimes \mathcal{H}_B, \rho)$



Open quantum systems

Time evolution: $|\psi_j(t)\rangle = e^{-iHt}|\psi_j(0)\rangle$

Hamiltonian:

$$H = H_S \otimes \mathbb{I}_E + \mathbb{I}_S \otimes H_E + H_I$$

Interaction picture:

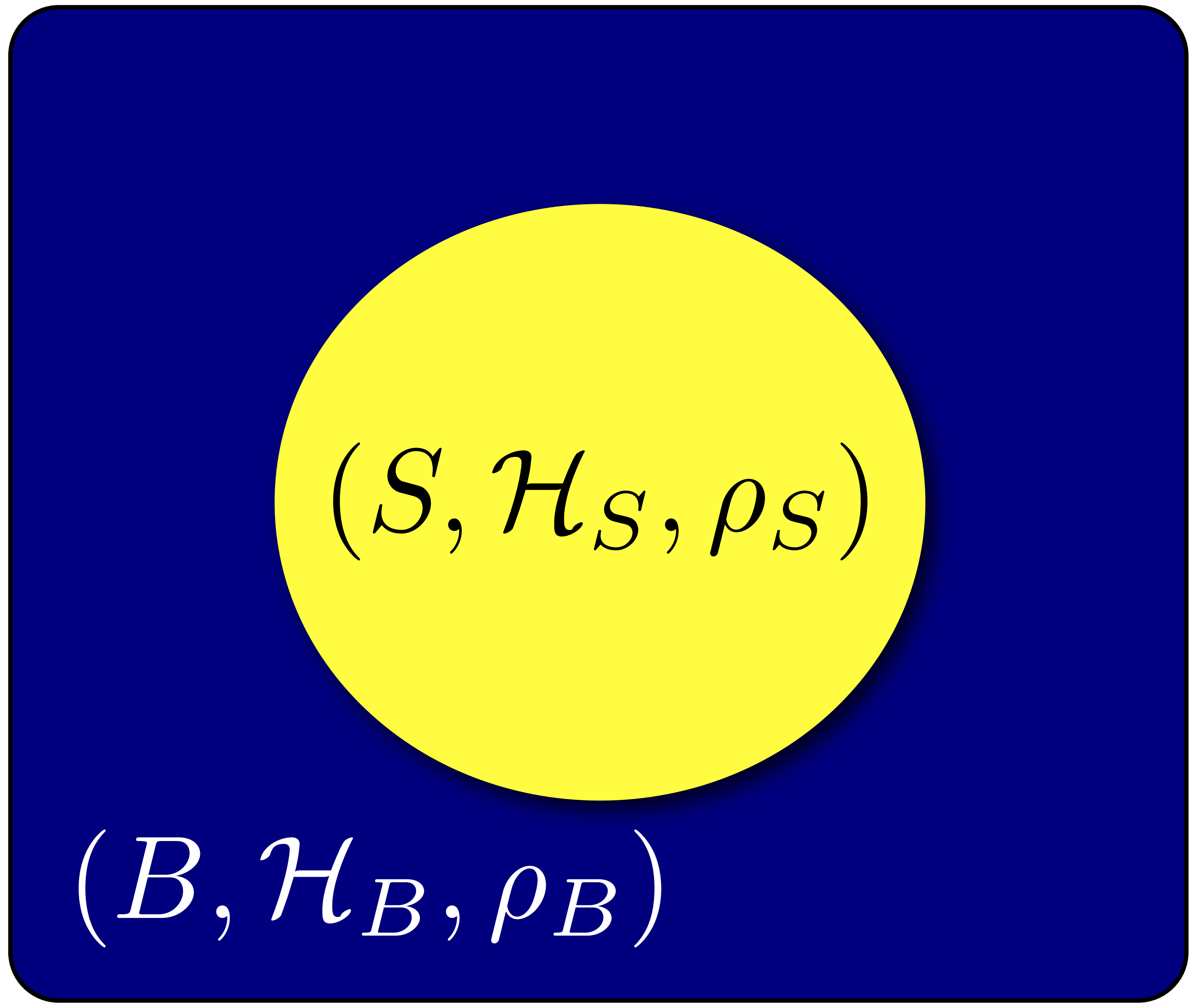
$$\tilde{\rho}(t) = U_I(t)\rho(0)U_I^\dagger(t) \qquad U_I(t) = \mathcal{T}e^{-i\int_0^t ds\, \tilde{H}_I(s)}$$

$$\tilde{H}_I(t) = e^{i(H_S+H_E)t}H_I e^{-i(H_S+H_E)t}$$

Von Neumann equation:

$$\frac{d}{dt}\tilde{\rho}(t) = -i[\tilde{H}_I(t), \tilde{\rho}(t)]$$

$(S+B, \mathcal{H}_S \otimes \mathcal{H}_B, \rho)$



Open quantum systems

Second-order master equation:

$$\frac{d}{dt}\tilde{\rho}(t) = -i[\tilde{H}_I(t), \tilde{\rho}(t)] \qquad \tilde{\rho}(t) = \rho(0) - i\int_0^t ds[\tilde{H}_I(s), \tilde{\rho}(s)]$$

$$\frac{d}{dt}\tilde{\rho}_S(t) = -\int_0^t d\tau \mathrm{Tr}_E[\tilde{H}_I(t), [\tilde{H}_I(t-\tau), \tilde{\rho}(t-\tau)]]$$

where $\mathrm{Tr}_E[\tilde{H}_I(t), \rho(0)] = 0$

Markovian master equation

Second-order master equation:

$$\frac{d}{dt}\tilde{\rho}(t) = -i[\tilde{H}_I(t), \tilde{\rho}(t)] \qquad \tilde{\rho}(t) = \rho(0) - i\int_0^t ds[\tilde{H}_I(s), \tilde{\rho}(s)]$$

$$\frac{d}{dt}\tilde{\rho}_S(t) = -\int_0^t d\tau \mathrm{Tr}_E[\tilde{H}_I(t), [\tilde{H}_I(t-\tau), \tilde{\rho}(t-\tau)]]$$

where $\mathrm{Tr}_E[\tilde{H}_I(t), \rho(0)] = 0$

Born (factorization) approximation: $\rho(t) \approx \rho_S(t) \otimes \rho_E$

$$\frac{d}{dt}\tilde{\rho}_S(t) = -\int_0^t d\tau \mathrm{Tr}_E[\tilde{H}_I(t), [\tilde{H}_I(t-\tau), \tilde{\rho}_S(t-\tau) \otimes \rho_E]]$$

Markovian master equation

Stellenbosch UNIVERSITY IYUNIVESITHI UNIVERSITEIT

Second-order master equation:

$$\frac{d}{dt}\tilde{\rho}_S(t) = -\int_0^t d\tau \mathrm{Tr}_E[\tilde{H}_I(t), [\tilde{H}_I(t-\tau), \tilde{\rho}_S(t-\tau) \otimes \rho_E]]$$

Markov approximation(s): $\tau_R \gg \tau_E$

where $\tau_R =$ System relaxation time

$\tau_E =$ Environment correlation time

(i) $\tilde{\rho}_S(t-\tau) \Longrightarrow \tilde{\rho}_S(t)$

(ii) $\int_0^t d\tau \ldots \Longrightarrow \int_0^\infty d\tau \ldots$

Markovian master equation

Stellenbosch UNIVERSITY IYUNIVESITHI UNIVERSITEIT

Second-order master equation:

$$\frac{d}{dt}\tilde{\rho}_S(t) = -\int_0^t d\tau \mathrm{Tr}_E[\tilde{H}_I(t), [\tilde{H}_I(t-\tau), \tilde{\rho}_S(t-\tau) \otimes \rho_E]]$$

Markov approximation(s): $\tau_R \gg \tau_E$

where $\tau_R =$ System relaxation time

$\tau_E =$ Environment correlation time

(i) $\tilde{\rho}_S(t-\tau) \Longrightarrow \tilde{\rho}_S(t)$

(ii) $\int_0^t d\tau \ldots \Longrightarrow \int_0^\infty d\tau \ldots$

Removal of memory effects

Markovian master equation

Stellenbosch UNIVERSITY IYUNIVESITHI UNIVERSITEIT

Second-order master equation:

$$\frac{d}{dt}\tilde{\rho}_S(t) = -\int_0^t d\tau \mathrm{Tr}_E[\tilde{H}_I(t), [\tilde{H}_I(t-\tau), \tilde{\rho}_S(t-\tau) \otimes \rho_E]]$$

Markov approximation(s): $\tau_R \gg \tau_E$

where $\tau_R =$ System relaxation time

$\tau_E =$ Environment correlation time

(i) $\tilde{\rho}_S(t-\tau) \Longrightarrow \tilde{\rho}_S(t)$

(ii) $\int_0^t d\tau \ldots \Longrightarrow \int_0^\infty d\tau \ldots$

Removal of memory effects

Markovian master equation

$$\frac{d}{dt}\tilde{\rho}_S(t) = -\int_0^\infty d\tau \mathrm{Tr}_E[\tilde{H}_I(t), [\tilde{H}_I(t-\tau), \tilde{\rho}_S(t) \otimes \rho_E]]$$

time-local

Markovian master equation

Stellenbosch UNIVERSITY IYUNIVESITHI UNIVERSITEIT

$$\frac{d}{dt}\tilde{\rho}_S(t) = -\int_0^\infty d\tau \mathrm{Tr}_E[\tilde{H}_I(t), [\tilde{H}_I(t-\tau), \tilde{\rho}_S(t) \otimes \rho_E]]$$

Interaction Hamiltonian: $\tilde{H}_I(t) = \sum_k \tilde{A}_k(t) \otimes \tilde{B}_k(t)$

$$A_k \in \mathcal{B}(\mathcal{H}_S)$$

$$B_k \in \mathcal{B}(\mathcal{H}_E)$$

$$\tilde{A}_k(t) = e^{iH_S t} A_k e^{-iH_S t}$$

$$\tilde{B}_k(t) = e^{iH_E t} B_k e^{-iH_E t}$$

Markovian master equation

Stellenbosch UNIVERSITY IYUNIVESITHI UNIVERSITEIT

$$\frac{d}{dt}\tilde{\rho}_S(t) = -\int_0^\infty d\tau \mathrm{Tr}_E[\tilde{H}_I(t), [\tilde{H}_I(t-\tau), \tilde{\rho}_S(t) \otimes \rho_E]]$$

Interaction Hamiltonian: $\tilde{H}_I(t) = \sum_k \tilde{A}_k(t) \otimes \tilde{B}_k(t)$

$$A_k \in \mathcal{B}(\mathcal{H}_S)$$

$$B_k \in \mathcal{B}(\mathcal{H}_E)$$

$$\tilde{A}_k(t) = e^{iH_S t} A_k e^{-iH_S t}$$

$$\tilde{B}_k(t) = e^{iH_E t} B_k e^{-iH_E t}$$

Projection operator: $\Pi(\varepsilon) = |\varepsilon\rangle\langle\varepsilon|$ $\quad H_S = \sum_\varepsilon \varepsilon|\varepsilon\rangle\langle\varepsilon|$ $\quad \mathcal{H}_S = \mathrm{span}(|\varepsilon\rangle)$

$$\Pi(\varepsilon)|\psi\rangle \to |\varepsilon\rangle$$

Markovian master equation

$$\frac{d}{dt}\tilde{\rho}_S(t) = -\int_0^\infty d\tau \mathrm{Tr}_E[\tilde{H}_I(t), [\tilde{H}_I(t-\tau), \tilde{\rho}_S(t) \otimes \rho_E]]$$

Interaction Hamiltonian: $\tilde{H}_I(t) = \sum_k \tilde{A}_k(t) \otimes \tilde{B}_k(t)$

$$A_k \in \mathcal{B}(\mathcal{H}_S)$$

$$B_k \in \mathcal{B}(\mathcal{H}_E)$$

$$\tilde{A}_k(t) = e^{iH_S t} A_k e^{-iH_S t}$$

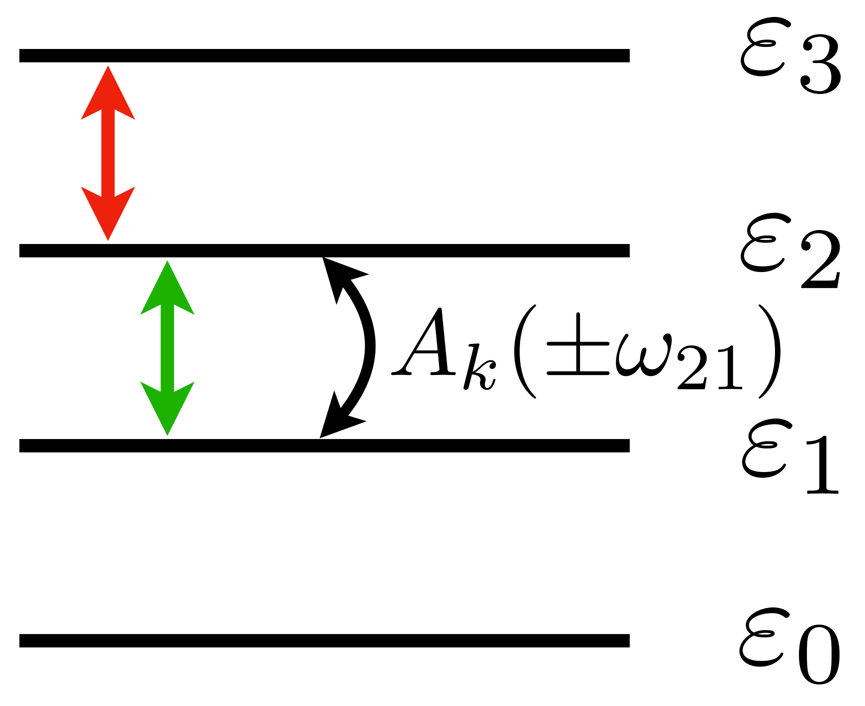
$$\tilde{B}_k(t) = e^{iH_E t} B_k e^{-iH_E t}$$

$$\boxed{A_k(\omega) = \sum_{\omega = \varepsilon' - \varepsilon} \Pi(\varepsilon) A_k \Pi(\varepsilon')}$$

$\omega =$ `energy difference'

$\omega_{32} = \varepsilon_3 - \varepsilon_2$

$\omega_{21} = \varepsilon_2 - \varepsilon_1$



Markovian master equation

Stellenbosch UNIVERSITY IYUNIVESITHI UNIVERSITEIT

$$\frac{d}{dt}\tilde{\rho}_S(t) = -\int_0^\infty d\tau \mathrm{Tr}_E[\tilde{H}_I(t), [\tilde{H}_I(t-\tau), \tilde{\rho}_S(t) \otimes \rho_E]]$$

Interaction Hamiltonian: $\tilde{H}_I(t) = \sum_k \tilde{A}_k(t) \otimes \tilde{B}_k(t)$

$$A_k \in \mathcal{B}(\mathcal{H}_S)$$

$$B_k \in \mathcal{B}(\mathcal{H}_E)$$

$$\tilde{A}_k(t) = e^{iH_S t} A_k e^{-iH_S t}$$

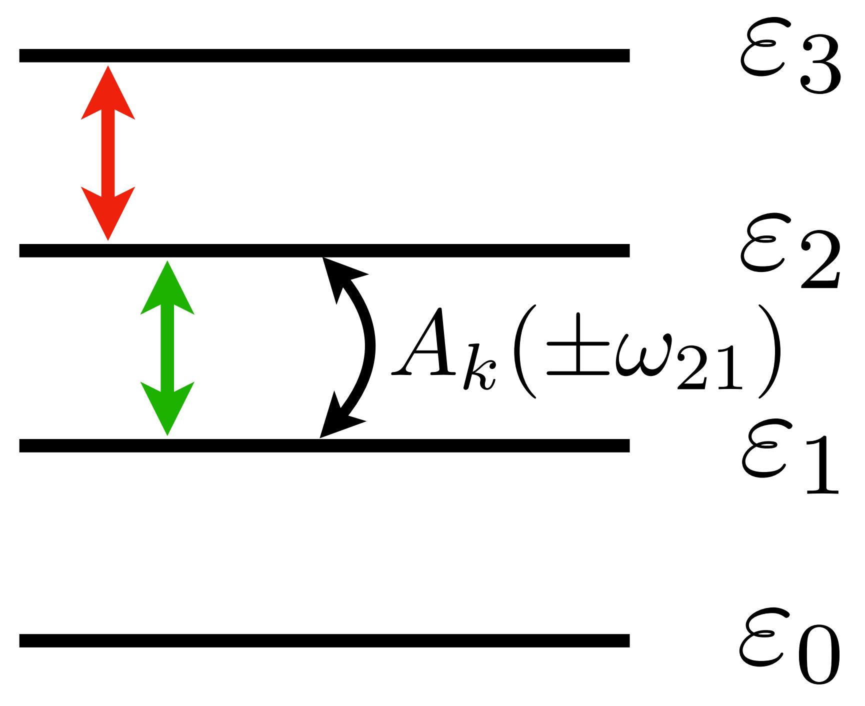
$$\tilde{B}_k(t) = e^{iH_E t} B_k e^{-iH_E t}$$

$\omega =$ `energy difference'

$$\boxed{A_k(\omega) = \sum_{\omega = \varepsilon' - \varepsilon} \Pi(\varepsilon) A_k \Pi(\varepsilon')}$$

$\omega_{32} = \varepsilon_3 - \varepsilon_2$

$\omega_{21} = \varepsilon_2 - \varepsilon_1$



$$[H_S, A_k(\omega)] = -\omega A_k(\omega)$$

Markovian master equation

Stellenbosch UNIVERSITY IYUNIVESITHI UNIVERSITEIT

$$\frac{d}{dt}\tilde{\rho}_S(t) = -\int_0^\infty d\tau \mathrm{Tr}_E[\tilde{H}_I(t), [\tilde{H}_I(t-\tau), \tilde{\rho}_S(t)\otimes\rho_E]]$$

Interaction Hamiltonian: $\tilde{H}_I(t) = \sum_k \tilde{A}_k(t)\otimes\tilde{B}_k(t)$

$$A_k \in \mathcal{B}(\mathcal{H}_S)$$

$$B_k \in \mathcal{B}(\mathcal{H}_E)$$

$$\tilde{A}_k(t) = e^{iH_St}A_k e^{-iH_St}$$

$$\tilde{B}_k(t) = e^{iH_Et}B_k e^{-iH_Et}$$

$$A_k(\omega) = \sum_{\omega=\varepsilon'-\varepsilon} \Pi(\varepsilon)A_k\Pi(\varepsilon')$$

$$[H_S, A_k(\omega)] = -\omega A_k(\omega) \Rightarrow e^{iH_St}A_k(\omega)e^{-iH_St} = e^{-i\omega t}A_k(\omega)$$

Markovian master equation

Stellenbosch UNIVERSITY IYUNIVESITHI UNIVERSITEIT

$$\frac{d}{dt}\tilde{\rho}_S(t) = -\int_0^\infty d\tau \mathrm{Tr}_E[\tilde{H}_I(t), [\tilde{H}_I(t-\tau), \tilde{\rho}_S(t)\otimes\rho_E]]$$

Interaction Hamiltonian: $\tilde{H}_I(t) = \sum_k \tilde{A}_k(t)\otimes\tilde{B}_k(t)$

$$A_k \in \mathcal{B}(\mathcal{H}_S)$$

$$B_k \in \mathcal{B}(\mathcal{H}_E)$$

$$\tilde{A}_k(t) = e^{iH_S t}A_k e^{-iH_S t}$$

$$\tilde{B}_k(t) = e^{iH_E t}B_k e^{-iH_E t}$$

$$A_k = \sum_\omega A_k(\omega) \quad \Rightarrow \quad \tilde{A}_k(t) = \sum_\omega A_k(\omega)e^{-i\omega t}$$

$$A_k(\omega) = \sum_{\omega=\varepsilon'-\varepsilon} \Pi(\varepsilon)A_k\Pi(\varepsilon')$$

$$[H_S, A_k(\omega)] = -\omega A_k(\omega) \quad \Rightarrow \quad e^{iH_S t}A_k(\omega)e^{-iH_S t} = e^{-i\omega t}A_k(\omega)$$

Markovian master equation

Stellenbosch UNIVERSITY IYUNIVESITHI UNIVERSITEIT

$$\frac{d}{dt}\tilde{\rho}_S(t) = -\int_0^\infty d\tau \mathrm{Tr}_E[\tilde{H}_I(t), [\tilde{H}_I(t-\tau), \tilde{\rho}_S(t) \otimes \rho_E]]$$

$$\tilde{H}_I(t) = \sum_k A_k(\omega) e^{-i\omega t} \otimes \tilde{B}_k(t) = \sum_k A_k^\dagger(\omega) e^{i\omega t} \otimes \tilde{B}_k(t)$$

$$\Rightarrow \quad \frac{d}{dt}\tilde{\rho}_S(t) = \sum_{k,l} \sum_{\omega,\omega' \in \Omega} e^{i(\omega'-\omega)t} \gamma_{kl}(\omega) \Big(A_l(\omega)\tilde{\rho}_S(t) A_k^\dagger(\omega') - A_k^\dagger(\omega') A_l(\omega) \tilde{\rho}_S(t) \Big) + \mathrm{h.c.}$$

Markovian master equation

Stellenbosch UNIVERSITY IYUNIVESITHI UNIVERSITEIT

$$\frac{d}{dt}\tilde{\rho}_S(t) = -\int_0^\infty d\tau \mathrm{Tr}_E[\tilde{H}_I(t), [\tilde{H}_I(t-\tau), \tilde{\rho}_S(t)\otimes\rho_E]]$$

$$\tilde{H}_I(t) = \sum_k A_k(\omega)e^{-i\omega t}\otimes\tilde{B}_k(t) = \sum_k A_k^\dagger(\omega)e^{i\omega t}\otimes\tilde{B}_k(t)$$

$$\frac{d}{dt}\tilde{\rho}_S(t) = \sum_{k,l}\sum_{\omega,\omega'\in\Omega} e^{i(\omega'-\omega)t}\gamma_{kl}(\omega)\Big(A_l(\omega)\tilde{\rho}_S(t)A_k^\dagger(\omega') - A_k^\dagger(\omega')A_l(\omega)\tilde{\rho}_S(t)\Big) + \text{h.c.}$$

$$\gamma_{kl}(\omega) = \int_{-\infty}^{\infty} d\tau\, C_{kl}(\tau)e^{i\omega\tau}$$

$$C_{kl}(\tau) = \mathrm{Tr}_E[\tilde{B}_k(\tau)B_l\rho_E(0)]$$

see Marco's lecture..

bath correlation function

Markovian master equation

$$\frac{d}{dt}\tilde{\rho}_S(t) = -\int_0^\infty d\tau \mathrm{Tr}_E[\tilde{H}_I(t), [\tilde{H}_I(t-\tau), \tilde{\rho}_S(t) \otimes \rho_E]]$$

$$\tilde{H}_I(t) = \sum_k A_k(\omega)e^{-i\omega t} \otimes \tilde{B}_k(t) = \sum_k A_k^\dagger(\omega)e^{i\omega t} \otimes \tilde{B}_k(t)$$

$$\frac{d}{dt}\tilde{\rho}_S(t) = \sum_{k,l} \sum_{\omega,\omega' \in \Omega} e^{i(\omega'-\omega)t}\gamma_{kl}(\omega)\Big(A_l(\omega)\tilde{\rho}_S(t)A_k^\dagger(\omega') - A_k^\dagger(\omega')A_l(\omega)\tilde{\rho}_S(t)\Big) + \mathrm{h.c.}$$

Secular approximation: (time scales)

$$\frac{d}{dt}\tilde{\rho}_S(t) = \sum_{k,l} \sum_{\omega} \gamma_{kl}(\omega)\Big(A_l(\omega)\tilde{\rho}_S(t)A_k^\dagger(\omega) - A_k^\dagger(\omega)A_l(\omega)\tilde{\rho}_S(t)\Big) + \mathrm{h.c.}$$

Markovian master equation

In the Schroedinger picture:

$$\frac{d}{dt}\tilde{\rho}_S(t) = \sum_{k,l}\sum_{\omega}\gamma_{kl}(\omega)\Big(A_l(\omega)\tilde{\rho}_S(t)A_k^\dagger(\omega) - A_k^\dagger(\omega)A_l(\omega)\tilde{\rho}_S(t)\Big) + \text{h.c.}$$

$$\Downarrow$$

$$\frac{d}{dt}\rho_S(t) = -i[H_S, \rho_S(t)] + \sum_{k,l}\sum_{\omega}\gamma_{kl}(\omega)\Big(A_l(\omega)\rho_S(t)A_k^\dagger(\omega) - \frac{1}{2}\{A_k^\dagger(\omega)A_l(\omega), \rho_S(t)\}\Big)$$

Hermitian $\boldsymbol{\gamma}(\omega) \succeq 0$:

$$\dot{\rho} = -i[H_S, \rho] + \sum_i \gamma_i\Big(L_i\rho L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, \rho\}\Big)$$

Markovian master equation

Stellenbosch UNIVERSITY IYUNIVESITHI UNIVERSITEIT

In the Schroedinger picture:

$$\frac{d}{dt}\tilde{\rho}_S(t) = \sum_{k,l}\sum_{\omega}\gamma_{kl}(\omega)\Big(A_l(\omega)\tilde{\rho}_S(t)A_k^\dagger(\omega) - A_k^\dagger(\omega)A_l(\omega)\tilde{\rho}_S(t)\Big) + \text{h.c.}$$

$$\frac{d}{dt}\rho_S(t) = -i[H_S,\rho_S(t)] + \sum_{k,l}\sum_{\omega}\gamma_{kl}(\omega)\Big(A_l(\omega)\rho_S(t)A_k^\dagger(\omega) - \frac{1}{2}\{A_k^\dagger(\omega)A_l(\omega),\rho_S(t)\}\Big)$$

Hermitian $\boldsymbol{\gamma}(\omega) \succeq 0$:

Markovian master equation

$$\dot{\rho} = -i[H_S,\rho] + \sum_i \gamma_i\Big(L_i\rho L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i,\rho\}\Big)$$

time-local

Time locality vs Markovianity

Stellenbosch UNIVERSITY IYUNIVESITHI UNIVERSITEIT

$$\frac{d}{dt}\rho_S(t) = -i[H_S, \rho_S(t)] - \int_0^t d\tau \mathcal{K}(\tau)\rho_S(t-\tau)$$

time-nonlocal

Markov approximation

$$\frac{d}{dt}\rho_S(t) = -i[H_S, \rho_S(t)] + \sum_i \gamma_i \Big(L_i \rho_S(t) L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, \rho_S(t)\}\Big)$$

time-local

Time locality vs Markovianity

Stellenbosch UNIVERSITY IYUNIVESITHI UNIVERSITEIT

$$\frac{d}{dt}\rho_S(t) = -i[H_S, \rho_S(t)] - \int_0^t d\tau \mathcal{K}(\tau)\rho_S(t-\tau)$$

time-nonlocal

Markov approximation

$$\frac{d}{dt}\rho_S(t) = -i[H_S, \rho_S(t)] + \sum_i \gamma_i \Big(L_i \rho_S(t) L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, \rho_S(t)\}\Big)$$

time-local

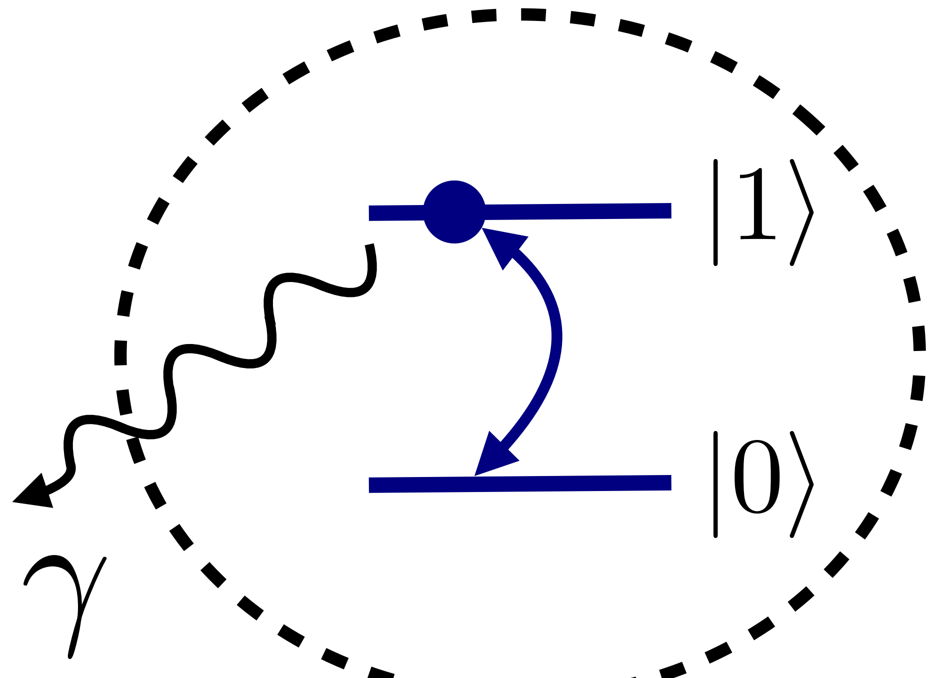
Tempting to make association of non-Markovianity with time-non locality..

memory effects $\stackrel{!}{=}$ time-nonlocal master equation

Example: Markovian master equation

$$\dot{\rho} = -i[H,\rho] + \sum_i \gamma_i \left(L_i \rho L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, \rho\} \right)$$

Spontaneous emission from qubit:



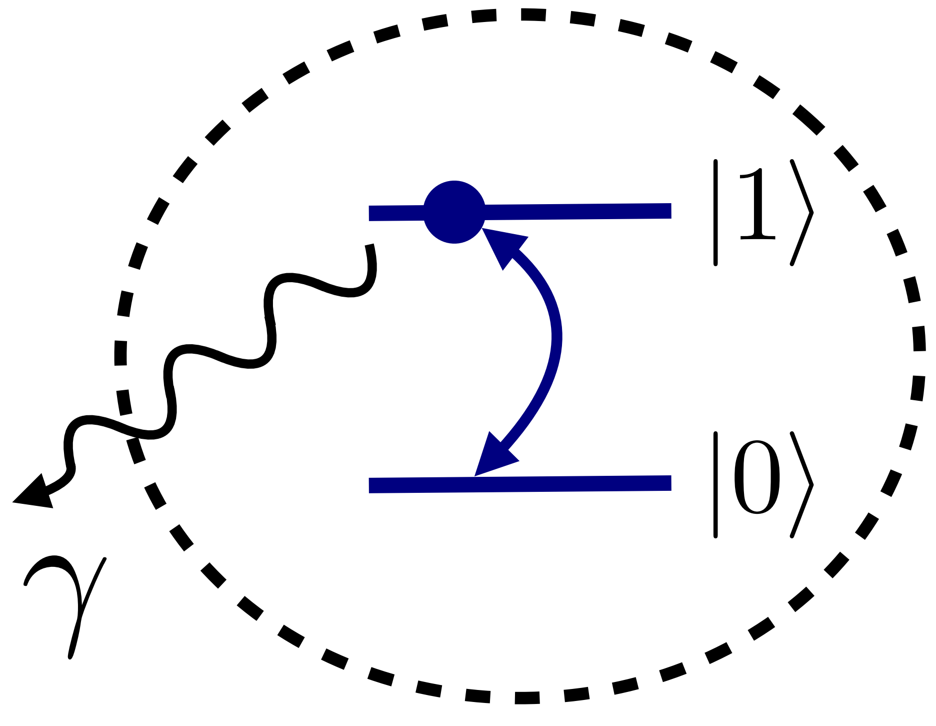
$$L = |0\rangle\langle 1|$$

$$H = \omega_0 |1\rangle\langle 1|$$

Example: Markovian master equation

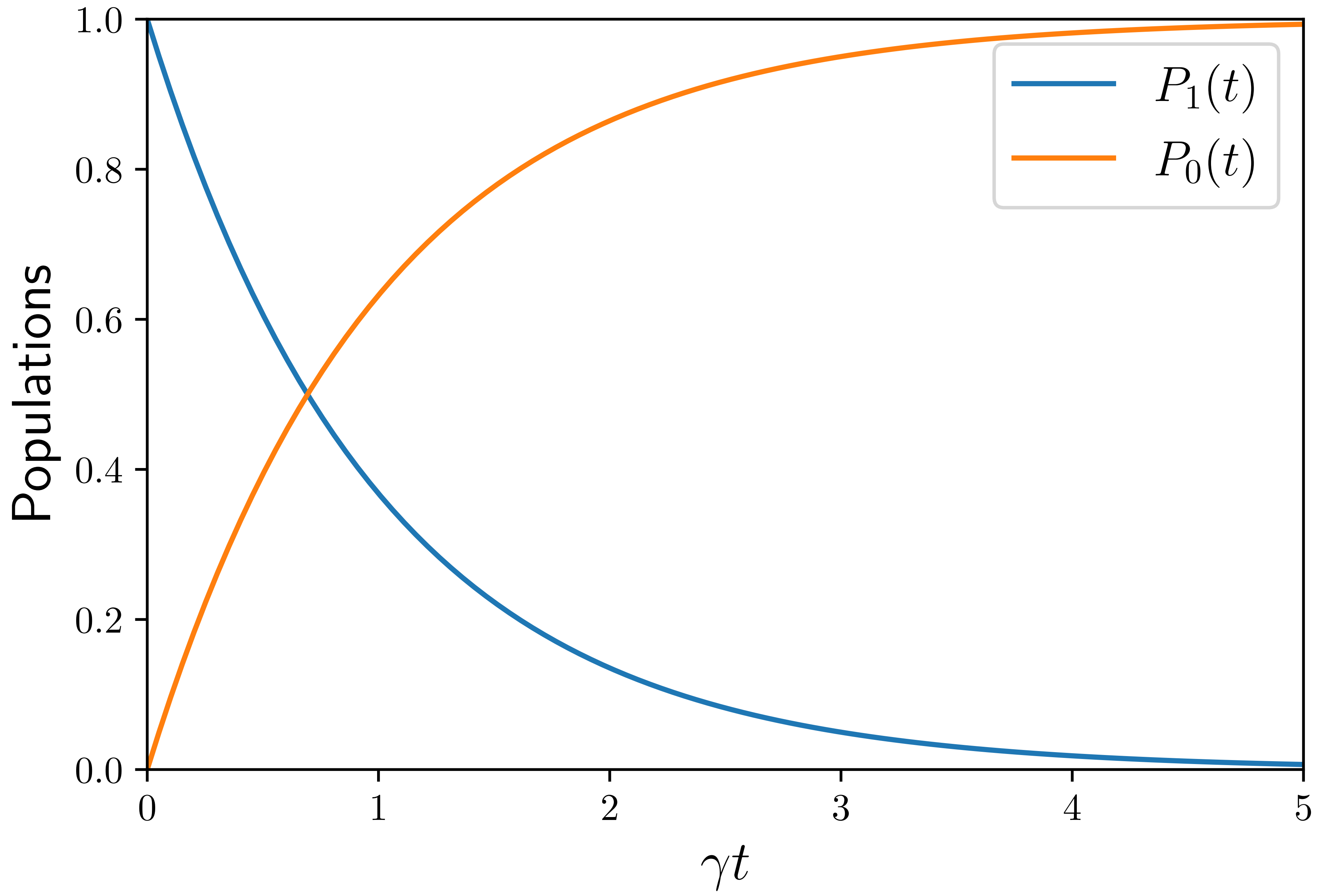
$$\dot{\rho} = -i[H,\rho] + \sum_i \gamma_i \Big(L_i \rho L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, \rho\} \Big)$$

Spontaneous emission from qubit:



$$L = |0\rangle\langle 1|$$

$$H = \omega_0 |1\rangle\langle 1|$$



Probability of emission event in interval $[t_i, t_i + \delta t]$

$$\delta p = \gamma \delta t \qquad (1-\delta p)^n = (1-\gamma\delta t)^{t/\delta t} \rightarrow e^{-\gamma t} \qquad \delta t = t/n$$

Semigroup structure

Stellenbosch
UNIVERSITY
IYUNIVESITHI
UNIVERSITEIT

$$\dot{\rho} = \mathcal{L}\rho = -i[H,\rho] + \sum_i \gamma_i \Big(L_i \rho L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, \rho\} \Big)$$

➡ $$\rho(t) = e^{\mathcal{L}t}\rho(0) = \Phi_t \rho(0)$$

$$\Phi_t : \mathcal{S}(\mathcal{H}_S) \longrightarrow \mathcal{S}(\mathcal{H}_S)$$

Semigroup structure

$$\dot{\rho} = \mathcal{L}\rho = -i[H,\rho] + \sum_i \gamma_i \left(L_i \rho L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, \rho\} \right)$$

$$\Rightarrow \quad \rho(t) = e^{\mathcal{L}t}\rho(0) = \Phi_t \rho(0) \qquad \Phi_t : \mathcal{S}(\mathcal{H}_S) \longrightarrow \mathcal{S}(\mathcal{H}_S)$$

Side note:

$$\Phi_t \rho = \mathrm{Tr}_E[e^{-iHt} \rho \otimes \rho_E e^{iHt}]$$

$$= \sum_\alpha K_\alpha(t) \rho K_\alpha^\dagger(t)$$

spectral decomposition of ρ_E

CPTP map

$$(\Phi \otimes \mathcal{I}_A^{(k)})\sigma \geq 0 \qquad \forall k \geq \dim \mathcal{H}_S$$

see Marco's lecture 3

Semigroup structure

Stellenbosch UNIVERSITY IYUNIVESITHI UNIVERSITEIT

$$\dot{\rho} = \mathcal{L}\rho = -i[H,\rho] + \sum_i \gamma_i \left(L_i \rho L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, \rho\} \right)$$

$$\Rightarrow \quad \rho(t) = e^{\mathcal{L}t}\rho(0) = \Phi_t \rho(0) \qquad \Phi_t : \mathcal{S}(\mathcal{H}_S) \longrightarrow \mathcal{S}(\mathcal{H}_S)$$

Markov semigroup:

$$\boxed{\Phi_{t+s} = \Phi_t \Phi_s \qquad \forall t, s \geq 0} \qquad e^{\mathcal{L}(t+s)} = e^{\mathcal{L}t} e^{\mathcal{L}s}$$

Semigroup structure

Stellenbosch UNIVERSITY IYUNIVESITHI UNIVERSITEIT

$$\dot{\rho} = \mathcal{L}\rho = -i[H,\rho] + \sum_i \gamma_i \left(L_i \rho L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, \rho\} \right)$$

$$\Rightarrow \quad \rho(t) = e^{\mathcal{L}t}\rho(0) = \Phi_t \rho(0) \qquad \Phi_t : \mathcal{S}(\mathcal{H}_S) \longrightarrow \mathcal{S}(\mathcal{H}_S)$$

Markov semigroup:

$$\Phi_{t+s} = \Phi_t \Phi_s \qquad \forall t,s \geq 0 \qquad\qquad e^{\mathcal{L}(t+s)} = e^{\mathcal{L}t} e^{\mathcal{L}s}$$

Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) theorem:

For a CPTP map Φ_t obeying the semigroup relation $\Phi_{t+s} = \Phi_t \Phi_s$:

$$\Updownarrow$$

$$\mathcal{L}\rho = -i[H,\rho] + \sum_{i=1}^{d^2-1} \gamma_i \left(L_i \rho L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, \rho\} \right) \qquad d = \dim \mathcal{H}_S$$

Semigroup structure

Stellenbosch UNIVERSITY IYUNIVESITHI UNIVERSITEIT

$$\dot{\rho} = \mathcal{L}\rho = -i[H,\rho] + \sum_i \gamma_i \Big(L_i \rho L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, \rho\} \Big)$$

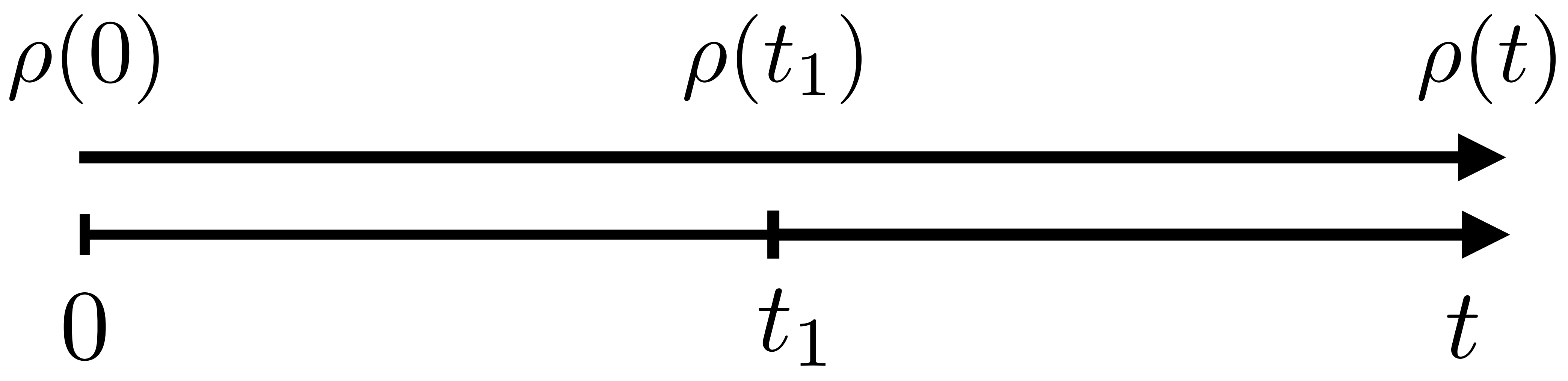
$$\Rightarrow \quad \rho(t) = e^{\mathcal{L}t}\rho(0) = \Phi_t \rho(0)$$

$$\Phi_t : \mathcal{S}(\mathcal{H}_S) \longrightarrow \mathcal{S}(\mathcal{H}_S)$$

Markov semigroup:

$$\Phi_{t+s} = \Phi_t \Phi_s \qquad \forall t, s \geq 0$$

memoryless



$$\Phi_t = \Phi_{t-t_1}\Phi_{t_1}$$

Summary - Markovian master equation

Stellenbosch
UNIVERSITY
IYUNIVESITHI
UNIVERSITEIT

$$\dot{\rho} = -i[H,\rho] + \sum_i \gamma_i \left(L_i \rho L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, \rho\} \right)$$

Born approximation: weak system-reservoir coupling $\tilde{\rho}(t) \approx \tilde{\rho}_S(t) \otimes \rho_E$

Markov approximation: fast reservoir timescale $\tau_E \ll \tau_R$

Secular approximation: fast (bare) system timescale $\tau_S \gg \tau_R$

Summary - Markovian master equation

Stellenbosch UNIVERSITY IYUNIVESITHI UNIVERSITEIT

$$\dot{\rho} = -i[H, \rho] + \sum_i \gamma_i \left(L_i \rho L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, \rho\} \right)$$

Born approximation: weak system-reservoir coupling $\tilde{\rho}(t) \approx \tilde{\rho}_S(t) \otimes \rho_E$

Markov approximation: fast reservoir timescale $\tau_E \ll \tau_R$

Secular approximation: fast (bare) system timescale $\tau_S \gg \tau_R$

Needed for complete positivity

time-local

Large separation of time scales between system and reservoir

Outline

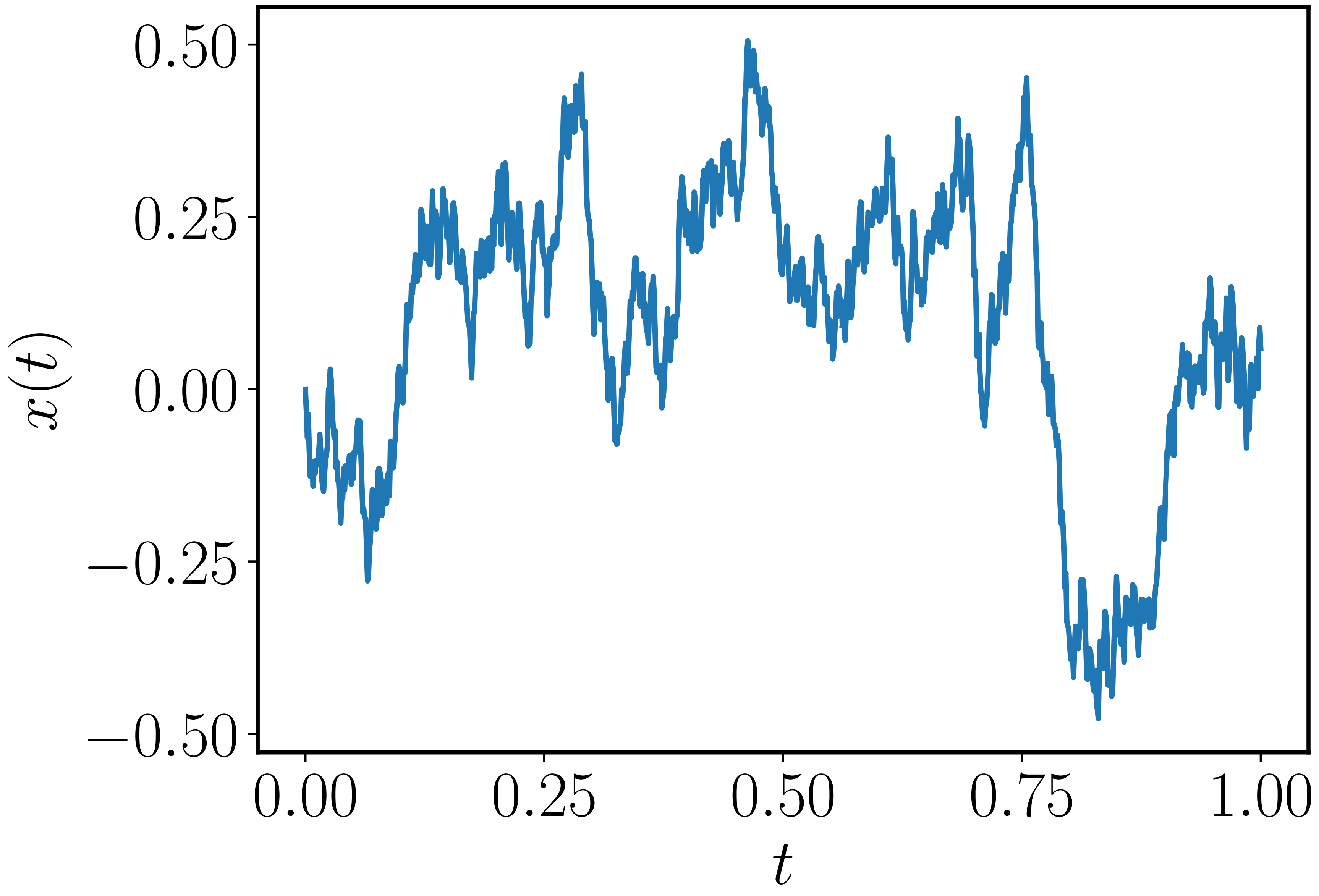
- Motivations.
- Open quantum systems - Markov approximation.
- Markovianity in classical stochastic processes.
- Quantum non-Markovianity - divisibility, distinguishability.
- Example: Spontaneous emission of two-level system.
- Collision models.

Classical stochastic processes

One-parameter family of random variables $\{X_t : t \in \mathcal{T}\}$ $X : \Omega \times \mathcal{T} \to \mathbb{R}$

Discrete time: $\mathcal{T} = \mathbb{N}^+$

Trajectory: $t \mapsto X(\omega, t), \quad t \in \mathcal{T}$



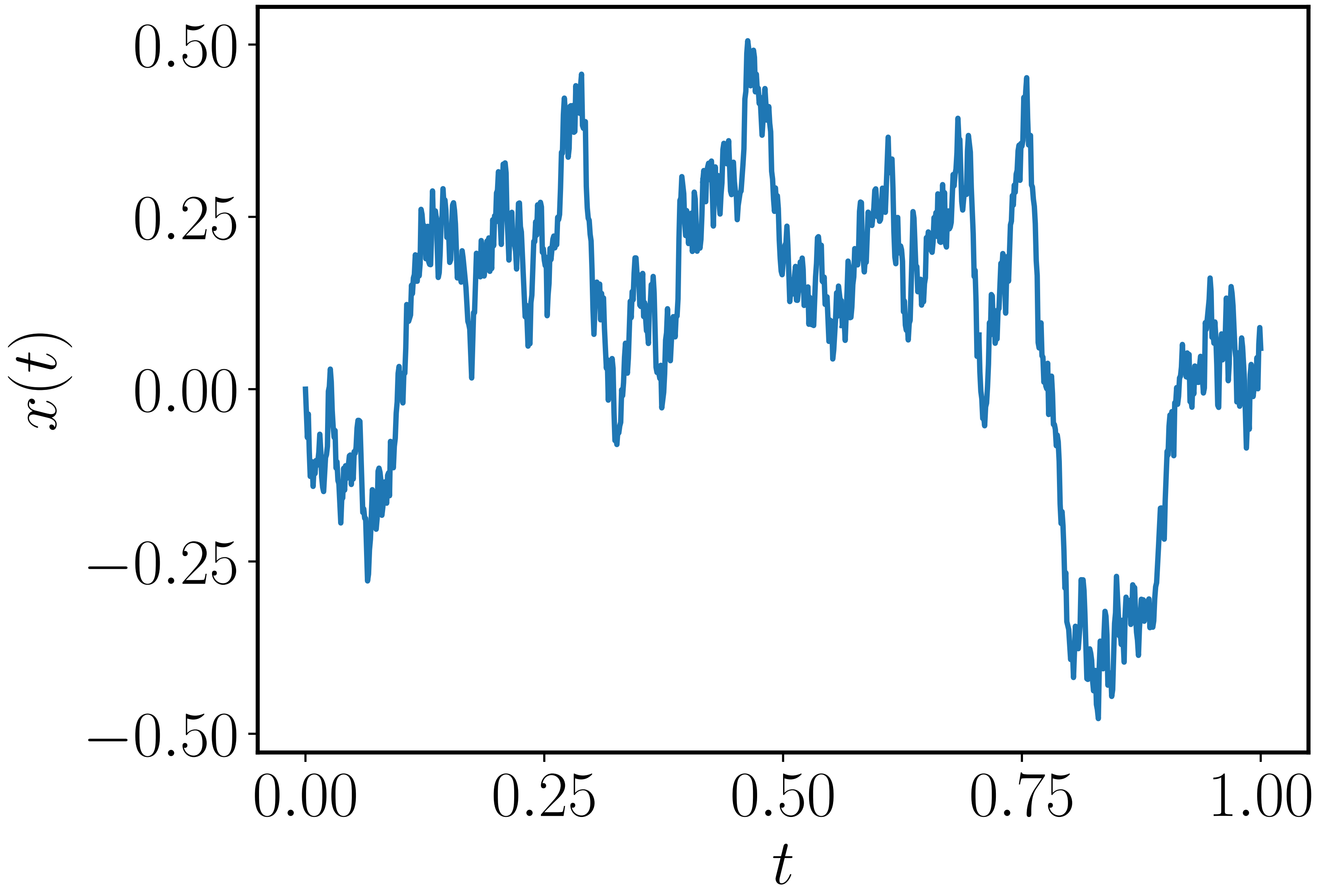
Classical stochastic processes

One-parameter family of random variables $\{X_t : t \in \mathcal{T}\}$ $X : \Omega \times \mathcal{T} \to \mathbb{R}$

Discrete time: $\mathcal{T} = \mathbb{N}^+$

Trajectory: $t \mapsto X(\omega, t), \quad t \in \mathcal{T}$

$$P(x_{t_n}, x_{t_{n-1}}, ..., x_{t_2}, x_{t_1})$$



gives probability of trajectory $x_{t_1} \to x_{t_2} \to ... \to x_{t_{n-1}} \to x_{t_n}$ $\quad t_n > t_{n-1} > ... > t_1$

Normalization: $$\sum_{x_{t_1},\ldots,x_{t_n}} P(x_{t_n}, ..., x_{t_1}) = 1 \qquad P(x_{t_n}, ..., x_{t_1}) \geq 0$$

Classical Markov processes

Stellenbosch UNIVERSITY IYUNIVESITHI UNIVERSITEIT

Conditional probability: probability of event $X = x_{t_n}$ given that

$$P(x_{t_n}|x_{t_n-1},\ldots,x_{t_1}) = \frac{P(x_{t_n},x_{t_{n-1}},\ldots,x_1)}{P(x_{t_n-1},\ldots,x_{t_1})}$$

Markov process:

$$P(x_{t_n}|x_{t_{n-1}},\ldots,x_{t_1}) = P(x_{t_n}|x_{t_{n-1}})$$

`Stochastic process with no memory'

$$\Rightarrow \quad P(x_{t_n},x_{t_{n-1}},\ldots,x_{t_1}) = P(x_{t_n}|x_{t_{n-1}})\ldots P(x_{t_2}|x_{t_1})P(x_{t_1})$$

Classical Markov processes

$$P(x_{t_3}, x_{t_2}, x_{t_1}) = P(x_{t_3}|x_{t_2}, x_{t_1})P(x_{t_2}, x_{t_1})$$

$$= P(x_{t_3}|x_{t_2})P(x_{t_2}, x_{t_1}) \quad \text{(Markov)}$$

$$t_3 > t_2 > t_1$$

$$\Rightarrow \quad P(x_{t_3}, x_{t_1}) = \sum_{x_{t_2}} P(x_{t_3}|x_{t_2})P(x_{t_2}, x_{t_1})$$

$$P(x_{t_{3,2}}|x_{t_1}) = \frac{P(x_{t_{3,2}}, x_{t_1})}{P(x_{t_1})}$$

Chapman-Kolmogorov equation:

$$\boxed{P(x_{t_3}|x_{t_1}) = \sum_{x_{t_2}} P(x_{t_3}|x_{t_2})P(x_{t_2}|x_{t_1})}$$

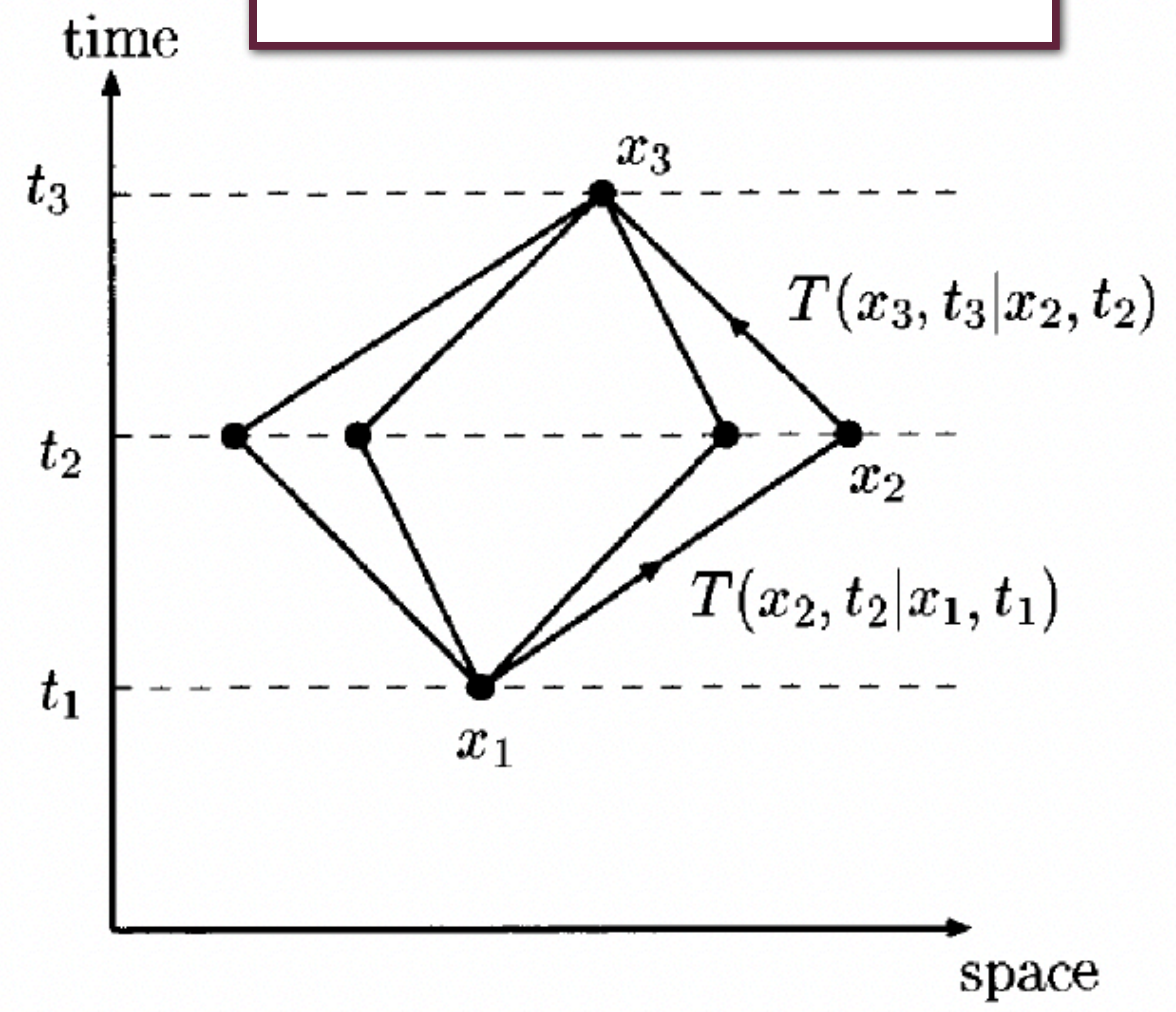
Classical Markov processes

$$P(x_{t_3}, x_{t_2}, x_{t_1}) = P(x_{t_3}|x_{t_2}, x_{t_1})P(x_{t_2}, x_{t_1}) \qquad t_3 > t_2 > t_1$$

$$= P(x_{t_3}|x_{t_2})P(x_{t_2}, x_{t_1}) \quad \text{(Markov)}$$

$$\Rightarrow \quad P(x_{t_3}, x_{t_1}) = \sum_{x_{t_2}} P(x_{t_3}|x_{t_2})P(x_{t_2}, x_{t_1})$$

$$T(x_t|y_s) = P(x_t|y_s)$$



Chapman-Kolmogorov equation:

$$P(x_{t_3}|x_{t_1}) = \sum_{x_{t_2}} P(x_{t_3}|x_{t_2})P(x_{t_2}|x_{t_1})$$

transition probability

Classical Markov processes

Stellenbosch UNIVERSITY IYUNIVESITHI UNIVERSITEIT

Classical Markov processes fully charararacterized by initial distribution $P(x_{t_0})$ and (conditional) transition probability $T(x_t|y_s) \equiv P(x_t|y_s)$, which obeys the Chapman-Kolmogorov equation

$$P(x_{t_n}, x_{t_{n-1}}, \dots, x_{t_1}) = \prod_{i=1}^{n-1} T(x_{t_{i+1}}|x_{t_i}) P(x_{t_1})$$

$$P(x_{t_1}) = \sum_{x_0} T(x_{t_1}|x_{t_0}) P(x_{t_0})$$

$$T(x_t|y_s) = \sum_z T(x_t|z_\tau) T(z_\tau|y_s)$$

Quantum Markov processes

Can we naturally extend the classical definition of a Markov process to the quantum domain?

$$P(x_{t_n}|x_{t_{n-1}}, ..., x_{t_1}) = P(x_{t_n}|x_{t_{n-1}})$$

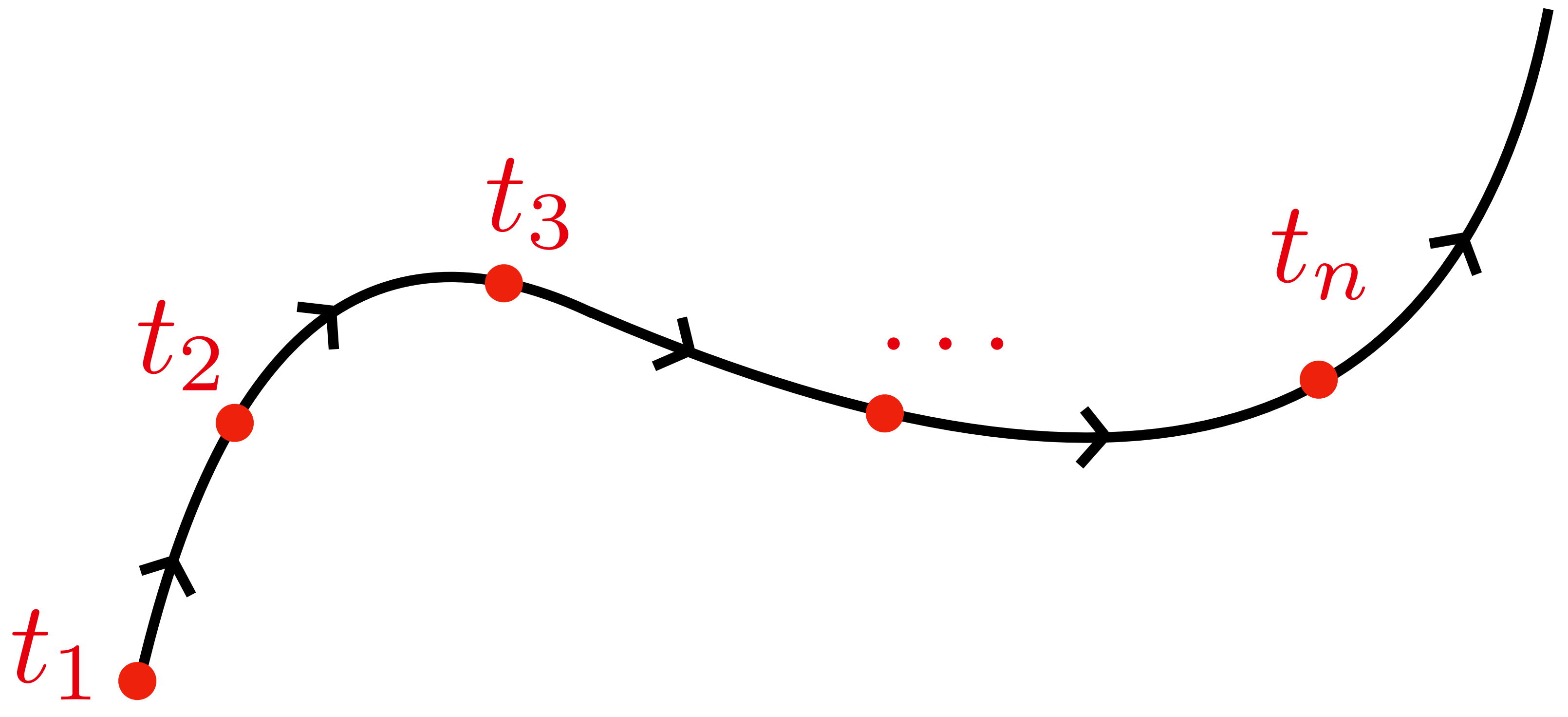
Quantum Markov processes

Can we naturally extend the classical definition of a Markov process to the quantum domain?

$$P(x_{t_n}|x_{t_{n-1}}, ..., x_{t_1}) = P(x_{t_n}|x_{t_{n-1}})$$

… not really

Measurements on a quantum system generally changes its behavior



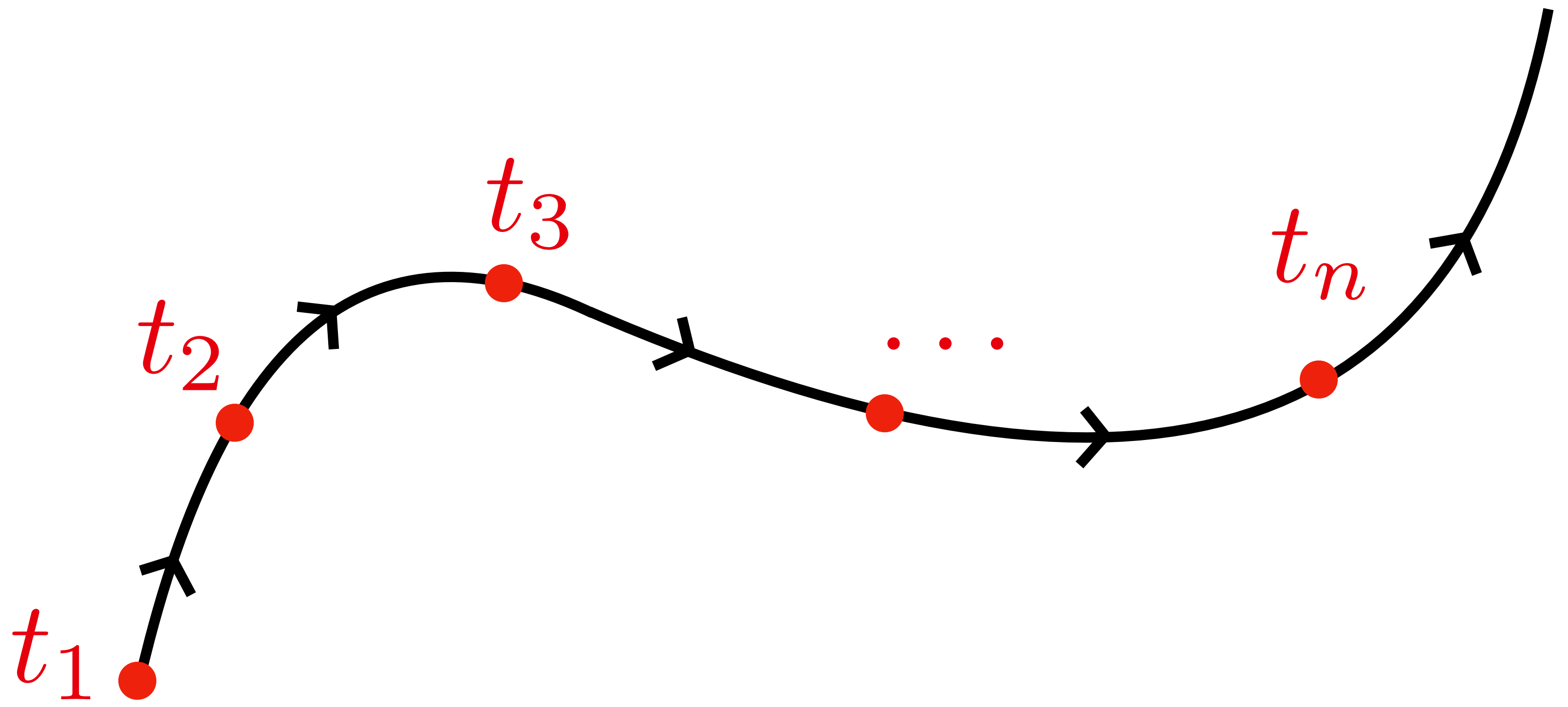
Quantum Markov processes

Can we naturally extend the classical definition of a Markov process to the quantum domain?

$$P(x_{t_n}|x_{t_{n-1}}, \ldots, x_{t_1}) = P(x_{t_n}|x_{t_{n-1}})$$

… not really

Measurements on a quantum system generally changes its behavior



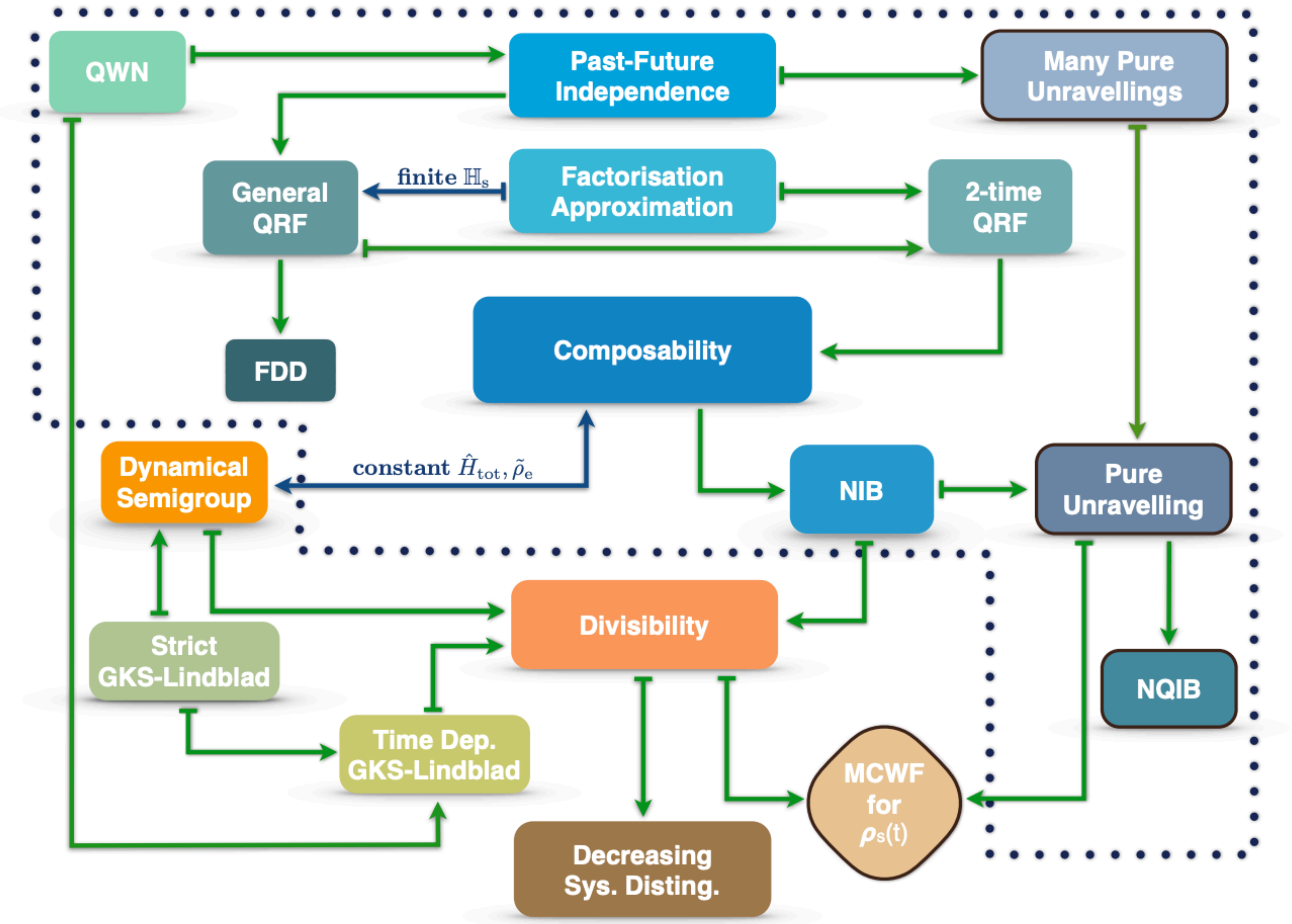
There is no quantum analogue of the $n-$point probability distribution $P(x_{t_n}, \ldots, x_{t_1})$

Approaches to quantum Markovianity

Global (extrinsic) picture: Markovianity is property of both the system and environment (e.g. past-future independence)

Reduced (intrinsic) picture: Markovianity property of the quantum dynamical map (e.g. divisibility)

F. Buscemi, 684th WE-Heraeus-Seminar, Bad Honnef, 04-12-18

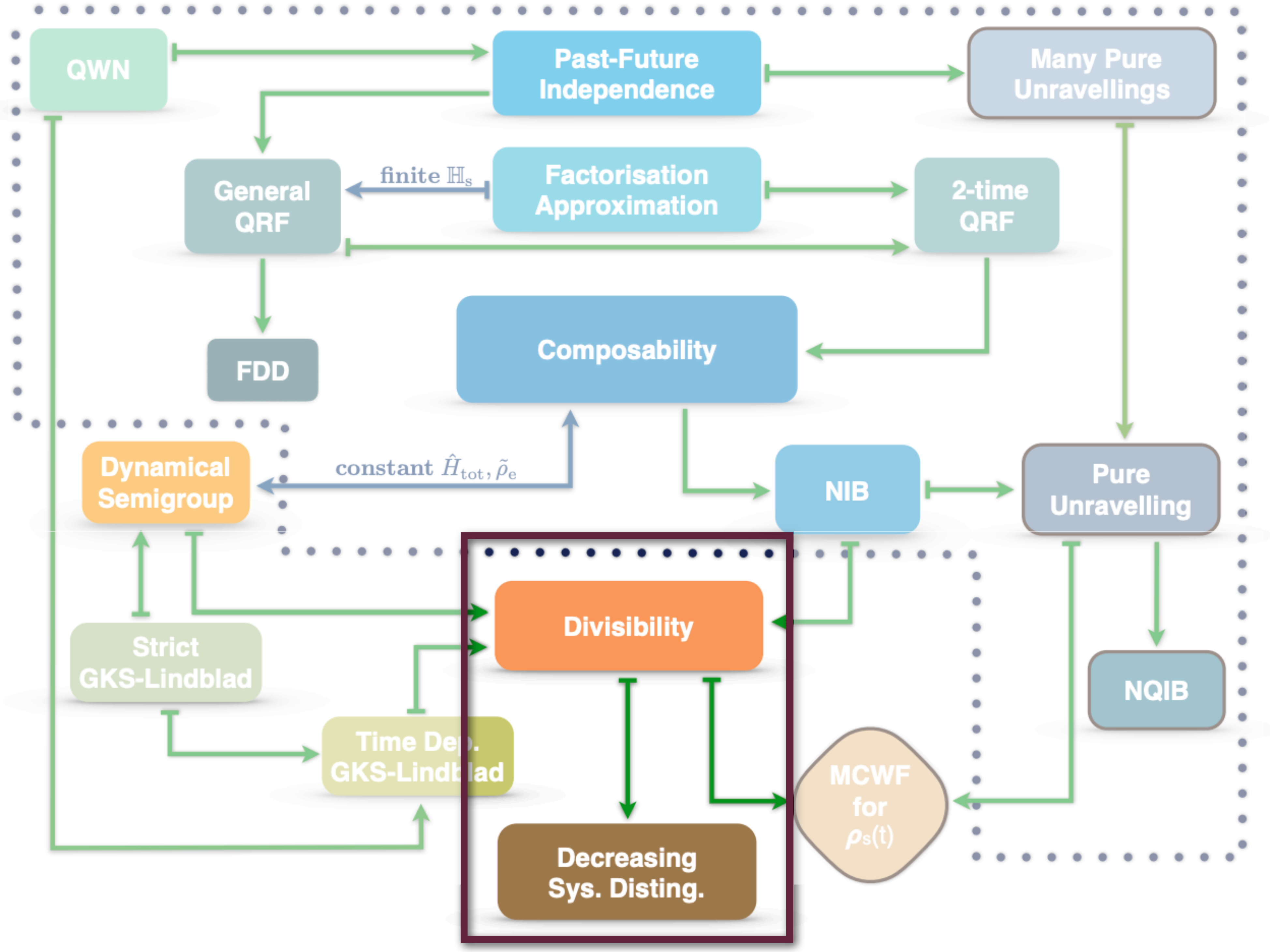


L. Li, M. J.W. Hall, & H. M. Wiseman, Phys. Rep. 759, 0370 (2018)

Approaches to quantum Markovianity

Reduced (intrinsic) picture: Markovianity property of the quantum dynamical map (e.g. divisibility)

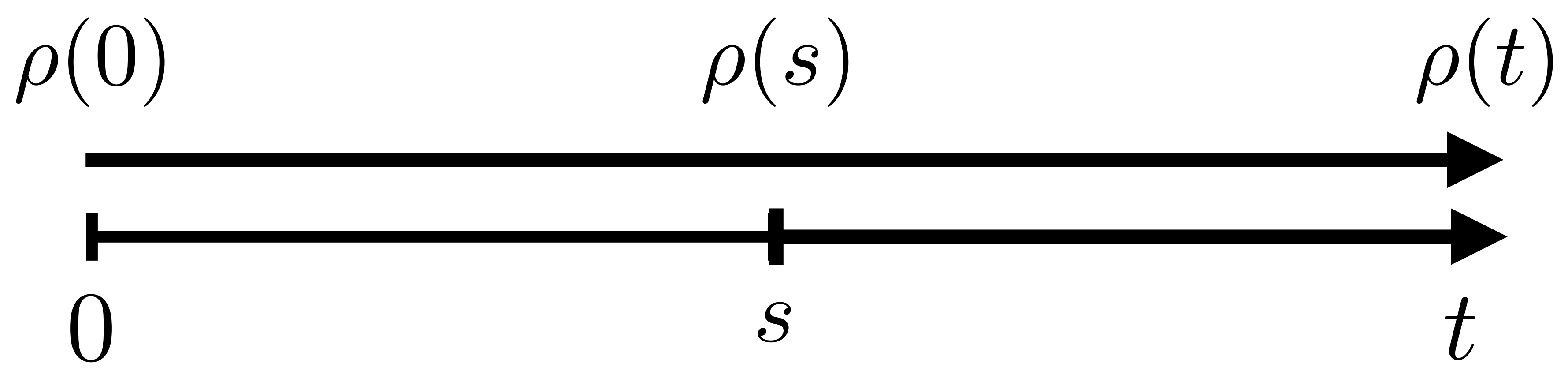
F. Buscemi, 684th WE-Heraeus-Seminar, Bad Honnef, 04-12-18



Li Li, Michael J.W. Hall, & Howard M. Wiseman, Phys. Rep. 759, 0370 (2018)

Divisibility

$$\Lambda_{t,s} = \Phi_t \Phi_s^{-1} \qquad t \geq s \geq 0$$



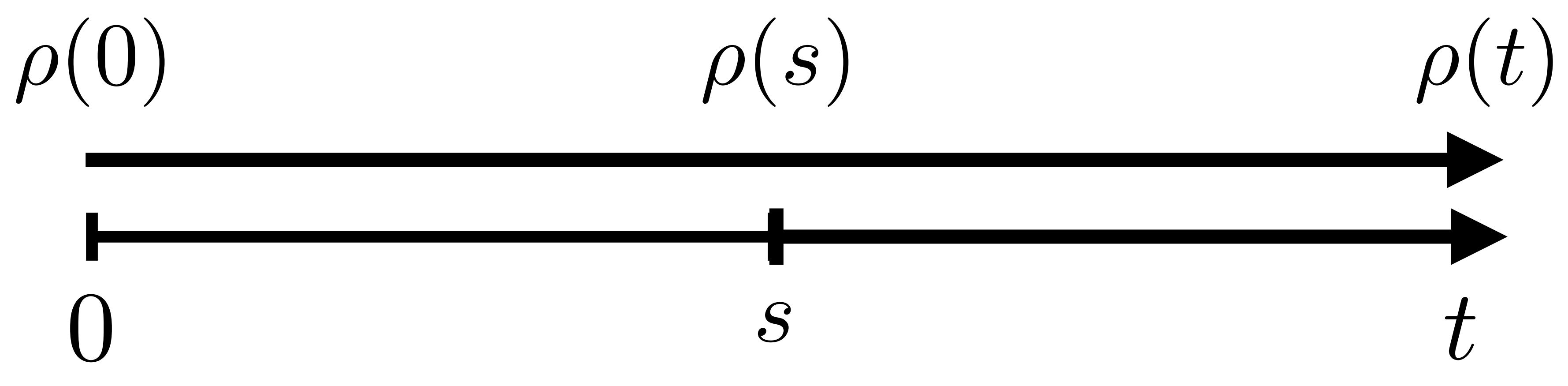
$\Lambda_{t,s}$ two-parameter map

$$\Rightarrow \quad \boxed{\Phi_t = \Lambda_{t,s}\Phi_s}$$

Stellenbosch
UNIVERSITY
IYUNIVESITHI
UNIVERSITEIT

Divisibility

$$\Lambda_{t,s} = \Phi_t \Phi_s^{-1} \qquad t \geq s \geq 0$$



$$\Rightarrow \quad \boxed{\Phi_t = \Lambda_{t,s} \Phi_s}$$

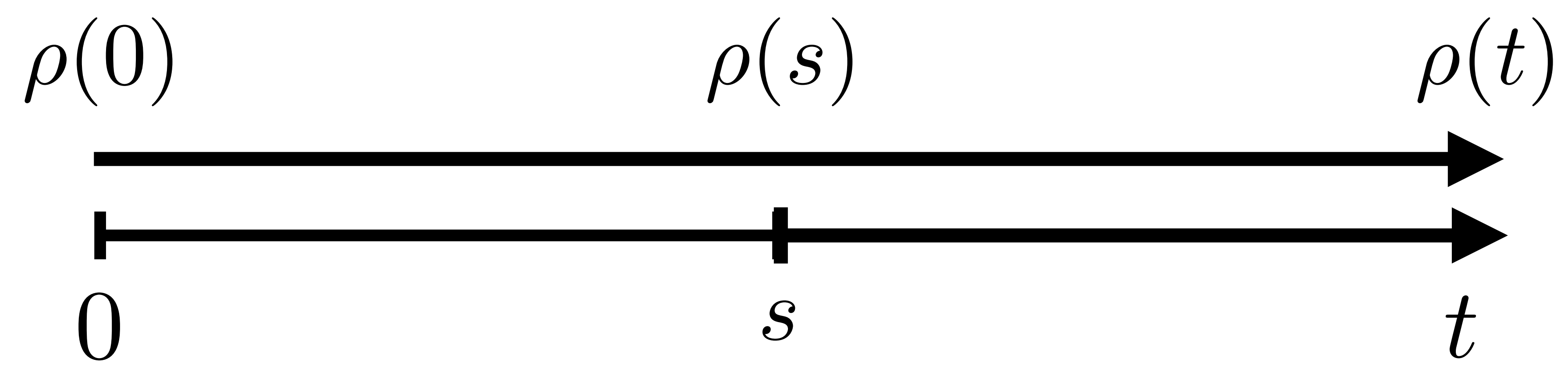
$\Lambda_{t,s}$ two-parameter map

- P-divisible if $\Lambda_{t,s}$ is a positive map $\rho \succeq 0, \; \Lambda_{t,s}\rho \succeq 0$.
- CP-divisible if $\Lambda_{t,s}$ is a completely positive map $\sigma \succeq 0, \, (\Phi \otimes \mathcal{I}_A^{(k)})\sigma \succeq 0$.

Divisibility

$$\Lambda_{t,s} = \Phi_t \Phi_s^{-1} \qquad t \geq s \geq 0$$

$$\Phi_t = \Lambda_{t,s} \Phi_s$$



$\Lambda_{t,s}$ two-parameter map

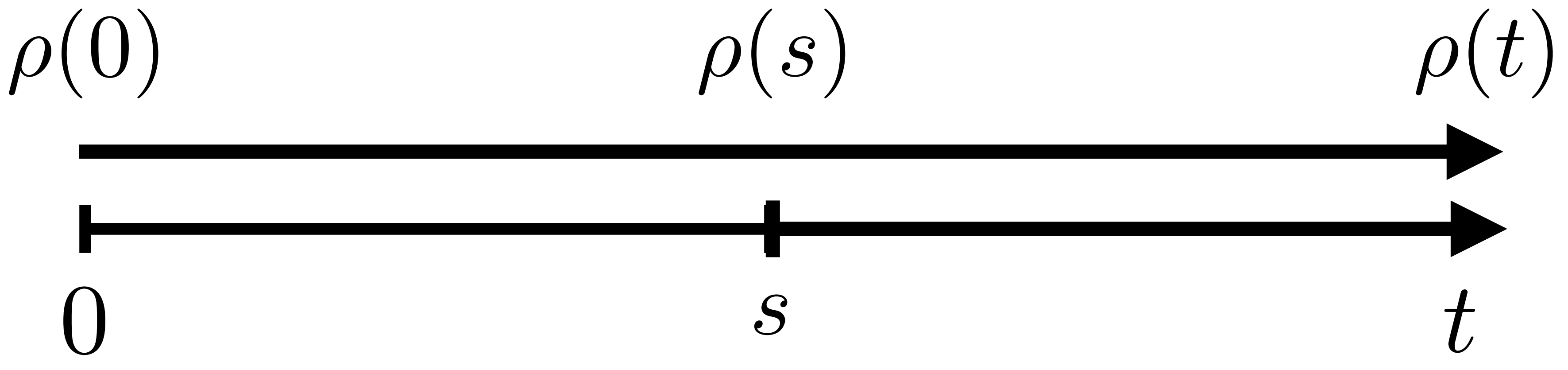
- P-divisible if $\Lambda_{t,s}$ is a positive map $\rho \succeq 0, \ \Lambda_{t,s}\rho \succeq 0.$
- CP-divisible if $\Lambda_{t,s}$ is a completely positive map $\sigma \succeq 0, \ (\Phi \otimes \mathcal{I}_A^{(k)})\sigma \succeq 0.$

Non-Markovian quantum process $=$ CP - indivisible

Divisibility

$$\Lambda_{t,s} = \Phi_t \Phi_s^{-1} \qquad t \geq s \geq 0$$

$$\Phi_t = \Lambda_{t,s} \Phi_s$$



$\Lambda_{t,s}$ two-parameter map

- P-divisible if $\Lambda_{t,s}$ is a positive map $\rho \succeq 0, \ \Lambda_{t,s}\rho \succeq 0.$
- CP-divisible if $\Lambda_{t,s}$ is a completely positive map $\sigma \succeq 0, \ (\Phi \otimes \mathcal{I}_A^{(k)})\sigma \succeq 0.$

Non-Markovian quantum process $=$ CP - indivisible

Markov semigroup: $\Lambda_{t,s} \Rightarrow e^{\mathcal{L}(t-s)} = \Phi_{t-s}$ CP-divisible

Divisibility

Stellenbosch UNIVERSITY IYUNIVESITHI UNIVERSITEIT

Existence of time-local master equation:

$$\Lambda_{t,s} = \Phi_t \Phi_s^{-1} \qquad t \geq s \geq 0$$

$$\frac{d}{dt}\rho_S(t) = \mathcal{L}_t \rho_S(t)$$

$$= -i[H_S(t), \rho_S(t)]$$

$$+ \sum_{i=1}^{d^2-1} \gamma_i(t) \Big(L_i(t)\rho_S(t)L_i^\dagger(t) - \frac{1}{2}\{L_i^\dagger(t)L_i(t), \rho_S(t)\}\Big)$$

time-local

$$\mathcal{L}_t = \lim_{\epsilon \to 0} \frac{\Lambda_{t+\epsilon,t} - \mathcal{I}}{\epsilon}$$

Divisibility

Existence of time-local master equation:

$$\Lambda_{t,s} = \Phi_t \Phi_s^{-1} \qquad t \geq s \geq 0$$

$$\frac{d}{dt}\rho_S(t) = \mathcal{L}_t \rho_S(t)$$

$$= -i[H_S(t), \rho_S(t)]$$

$$+ \sum_{i=1}^{d^2-1} \gamma_i(t) \Big(L_i(t)\rho_S(t)L_i^\dagger(t) - \frac{1}{2}\{L_i^\dagger(t)L_i(t), \rho_S(t)\}\Big)$$

time-local

$$\mathcal{L}_t = \lim_{\epsilon \to 0} \frac{\Lambda_{t+\epsilon,t} - \mathcal{I}}{\epsilon}$$

Time-independent: $\gamma \succeq 0 \quad H_S(t) = H_S$ ➡ $\Lambda_{t,s} = \Lambda_{t-s} = e^{\mathcal{L}(t-s)}$

Divisibility

Existence of time-local master equation:

$$\Lambda_{t,s} = \Phi_t \Phi_s^{-1} \qquad t \geq s \geq 0$$

$$\frac{d}{dt}\rho_S(t) = \mathcal{L}_t \rho_S(t)$$
$$= -i[H_S(t), \rho_S(t)]$$
$$+ \sum_{i=1}^{d^2-1} \gamma_i(t)\Big(L_i(t)\rho_S(t)L_i^\dagger(t) - \frac{1}{2}\{L_i^\dagger(t)L_i(t), \rho_S(t)\}\Big)$$

time-local

$$\mathcal{L}_t = \lim_{\epsilon \to 0} \frac{\Lambda_{t+\epsilon,t} - \mathcal{I}}{\epsilon}$$

Time-independent: $\gamma \succeq 0 \quad H_S(t) = H_S$ ➡ $\Lambda_{t,s} = \Lambda_{t-s} = e^{\mathcal{L}(t-s)}$

Time-dependent: $\mathrm{diag}(\gamma_1(t), ..., \gamma_{d^2-1}(t)) \succeq 0$ ➡ $\Lambda_{t,s}$ = CP-divisible

$\sum_i \gamma_i(t) |\langle n|A_i(t)|m\rangle|^2 \geq 0$ ➡ $\Lambda_{t,s}$ = P-divisible

Divisibility

Check complete positivity of intermediate map:

$$C_{\Lambda_{t,s}} = (\Lambda_{t,s} \otimes \mathcal{I}_A^{(d)})|\Psi\rangle\langle\Psi|$$

Choi matrix

$$|\Psi\rangle = \frac{1}{\sqrt{d}} \sum_{n=1}^{d} |n\rangle_S |n\rangle_A$$

maximally entangled state of system S and ancilla A.

A Rivas et al, PRL 105 050403 (2010)

Divisibility

Stellenbosch UNIVERSITY IYUNIVESITHI UNIVERSITEIT

Check complete positivity of intermediate map:

$$C_{\Lambda_{t,s}} = (\Lambda_{t,s} \otimes \mathcal{I}_A^{(d)})|\Psi\rangle\langle\Psi|$$

Choi matrix

$$|\Psi\rangle = \frac{1}{\sqrt{d}} \sum_{n=1}^{d} |n\rangle_S |n\rangle_A$$

maximally entangled state of system S and ancilla A.

$\Lambda_{t,s}$ is completely positive iff $C_{\Lambda_{t,s}} \succeq 0$

$\|C_{\Lambda_{t,s}}\|_1 = 1$ iff $\Lambda_{t,s}$ is completely positive

$$\|X\|_1 = \mathrm{Tr}\sqrt{X^\dagger X}$$

$$g(t) = \lim_{\epsilon \to 0+} \frac{\|C_{\Lambda_{t+\varepsilon,t}}\|_1 - 1}{\varepsilon}$$

$g(t) > 0$ non-Markovian

$g(t) = 0$ Markovian

A. Rivas et al, PRL 105 050403 (2010)

Distinguishability

Stellenbosch UNIVERSITY IYUNIVESITHI UNIVERSITEIT

Trace distance:

$$D(\rho,\sigma) = \frac{1}{2}\|\rho-\sigma\|_1 = \frac{1}{2}\mathrm{Tr}[(\rho-\sigma)^\dagger(\rho-\sigma)]$$

$$D(\rho,\sigma) = 0 \quad \Rightarrow \quad \rho = \sigma$$

$$D(\rho,\sigma) = 1 \quad \Rightarrow \quad \rho \perp \sigma$$

$$0 \leq D(\rho,\sigma) \leq 1$$

Distinguishability

Stellenbosch UNIVERSITY IYUNIVESITHI UNIVERSITEIT

Trace distance:

$$D(\rho,\sigma) = \frac{1}{2}\|\rho-\sigma\|_1 = \frac{1}{2}\mathrm{Tr}[(\rho-\sigma)^\dagger(\rho-\sigma)]$$

$$D(\rho,\sigma) = 0 \quad \Rightarrow \quad \rho = \sigma$$

$$0 \leq D(\rho,\sigma) \leq 1$$

$$D(\rho,\sigma) = 1 \quad \Rightarrow \quad \rho \perp \sigma$$

$$\Rightarrow \quad P_{\max} = \frac{1}{2}[1 + D(\rho,\sigma)]$$

maximum likelihood of distinguishing states ρ and σ with single shot measurement

Distinguishability

Stellenbosch UNIVERSITY IYUNIVESITHI UNIVERSITEIT

Trace distance:

$$D(\rho,\sigma) = \frac{1}{2}\|\rho - \sigma\|_1 = \frac{1}{2}\mathrm{Tr}[(\rho-\sigma)^\dagger(\rho-\sigma)]$$

$$D(\rho,\sigma) = 0 \quad \Rightarrow \quad \rho = \sigma \qquad\qquad 0 \leq D(\rho,\sigma) \leq 1$$

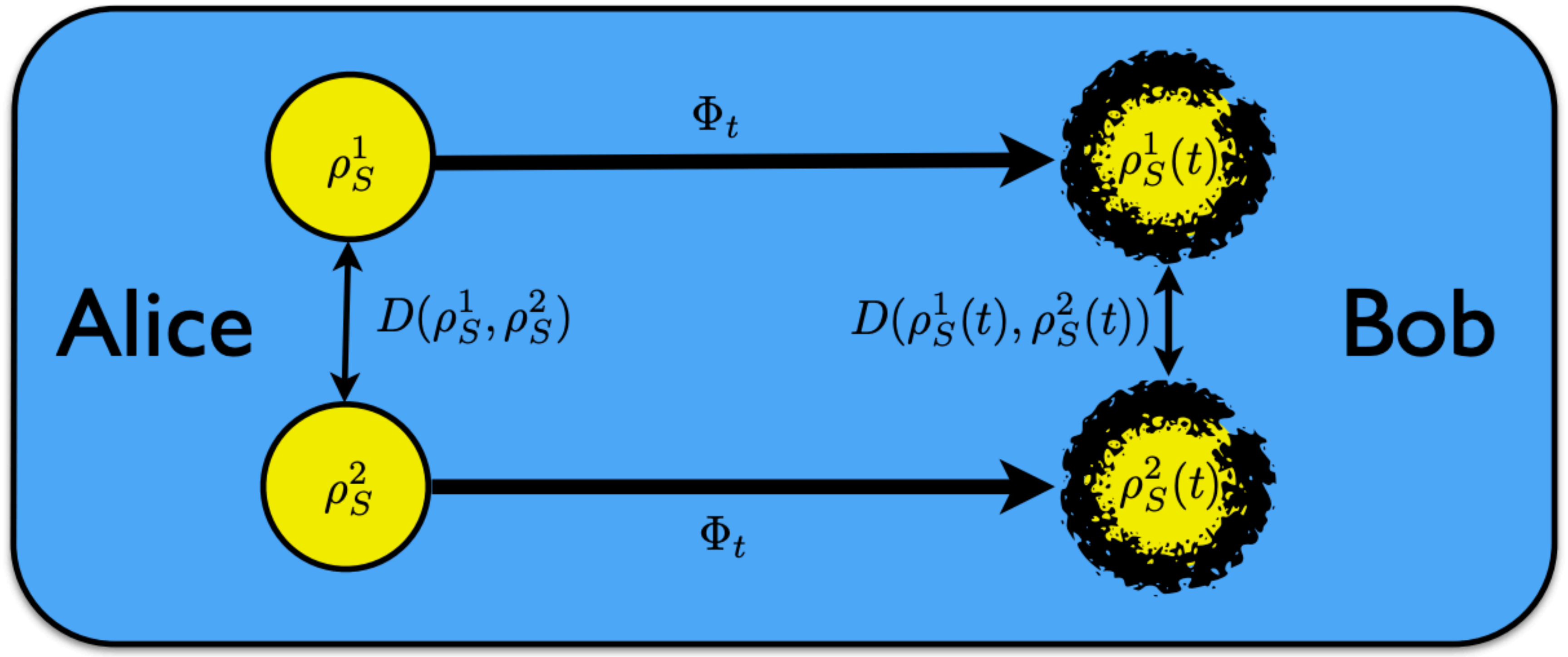
$$D(\rho,\sigma) = 1 \quad \Rightarrow \quad \rho \perp \sigma$$

$$P_{\max} = \frac{1}{2}[1 + D(\rho,\sigma)]$$

maximum likelihood of distinguishing states ρ and σ with single shot measurement

Monotonicity under CPTP maps:

$$D(\rho,\sigma) \geq D(\Phi_t\rho, \Phi_t\sigma)$$



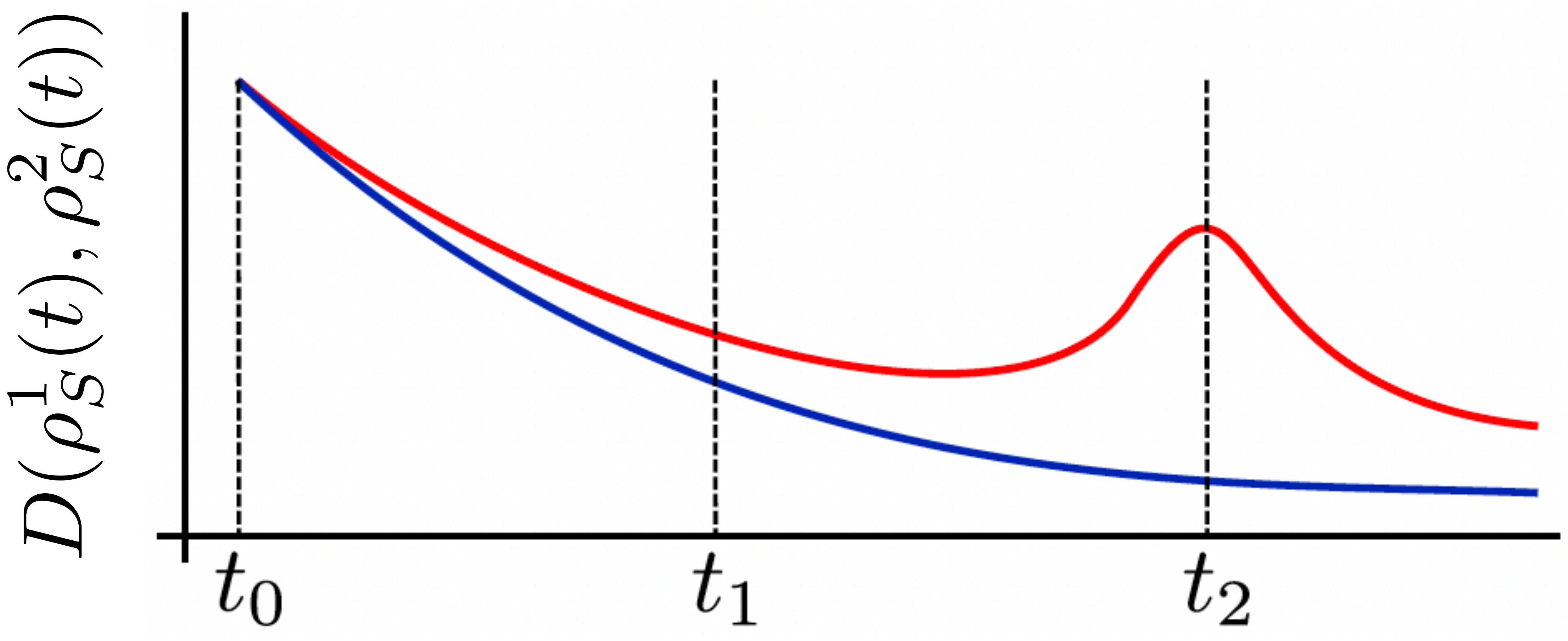
H.-P. Breuer et al RMP 88, 21002 (2016)

Distinguishability

Quantum process Φ_t non-Markovian if there is some pair of initial states $\{\rho_S^1(0), \rho_S^2(0)\}$ such that

$$D(\rho_S^1(t_2), \rho_S^2(t_2)) \geq D(\rho_S^1(t_1), \rho_S^2(t_1))$$

$$t_2 > t_1$$



H.-P. Breuer et al PRL 103, 210401 (2009)

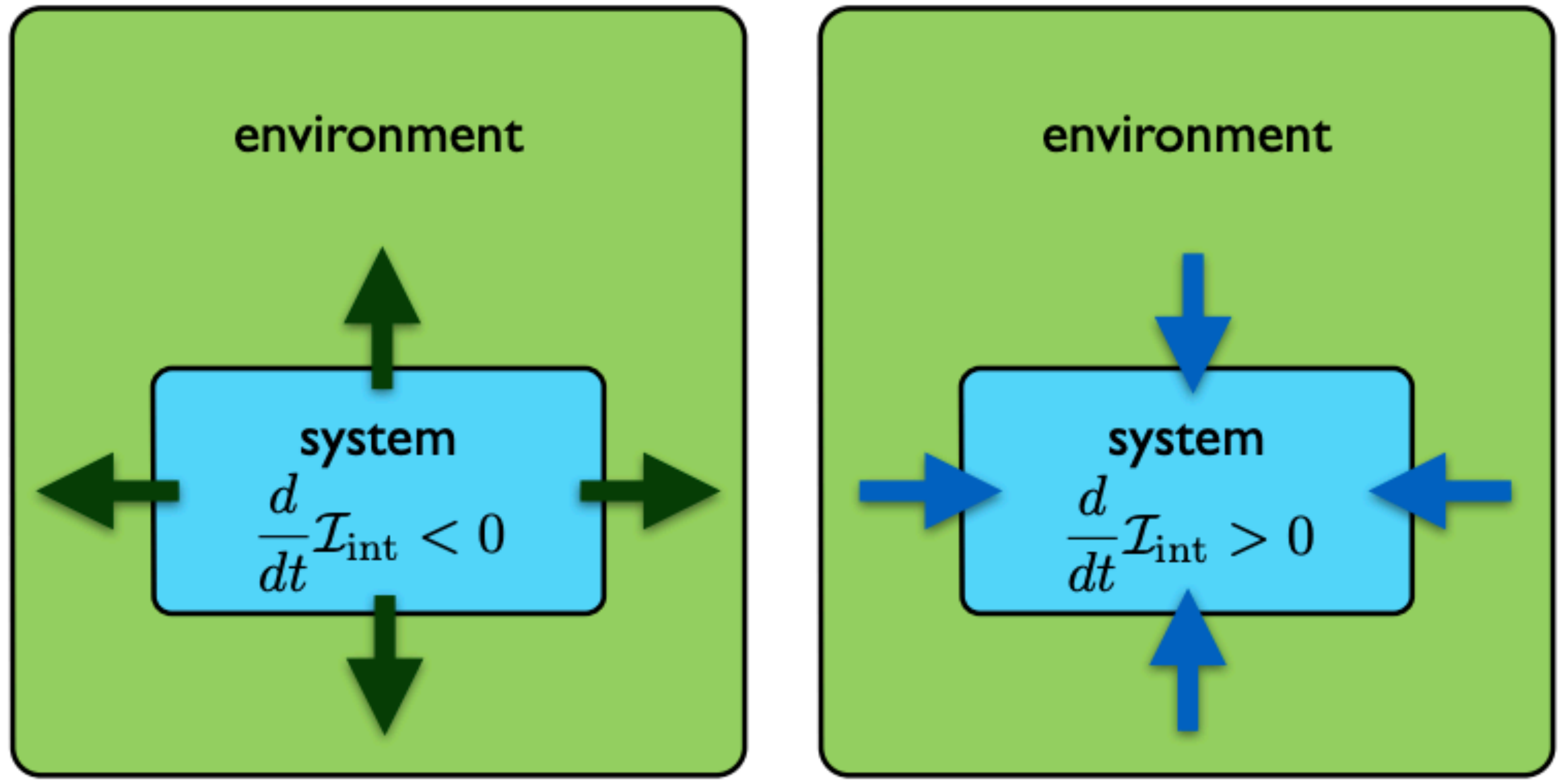
Distinguishability

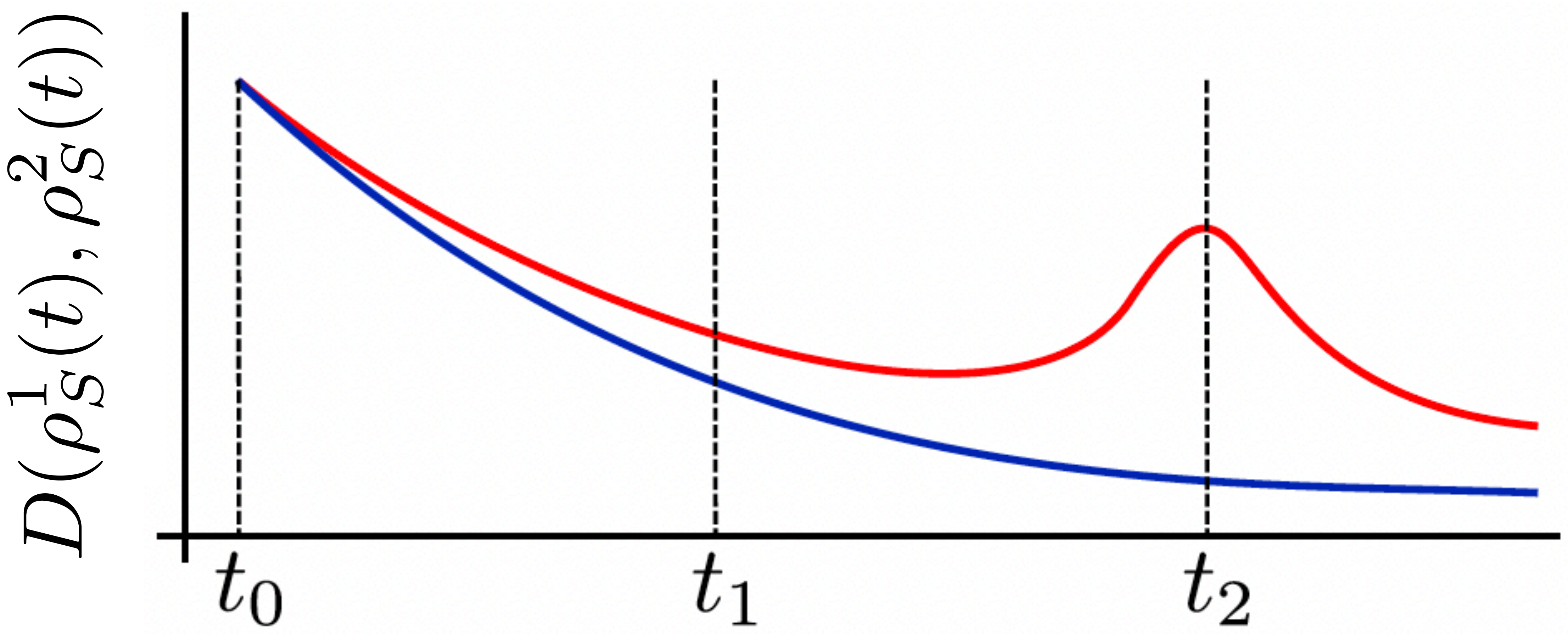
Quantum process Φ_t non-Markovian if there is some pair of initial states $\{\rho_S^1(0), \rho_S^2(0)\}$ such that

$$D(\rho_S^1(t_2), \rho_S^2(t_2)) \geq D(\rho_S^1(t_1), \rho_S^2(t_1))$$

$t_2 > t_1$

$$\mathcal{I}_{\text{int}} = D(\rho_S^1(t), \rho_S^2(t))$$





H.-P. Breuer et al PRL 103, 210401 (2009)

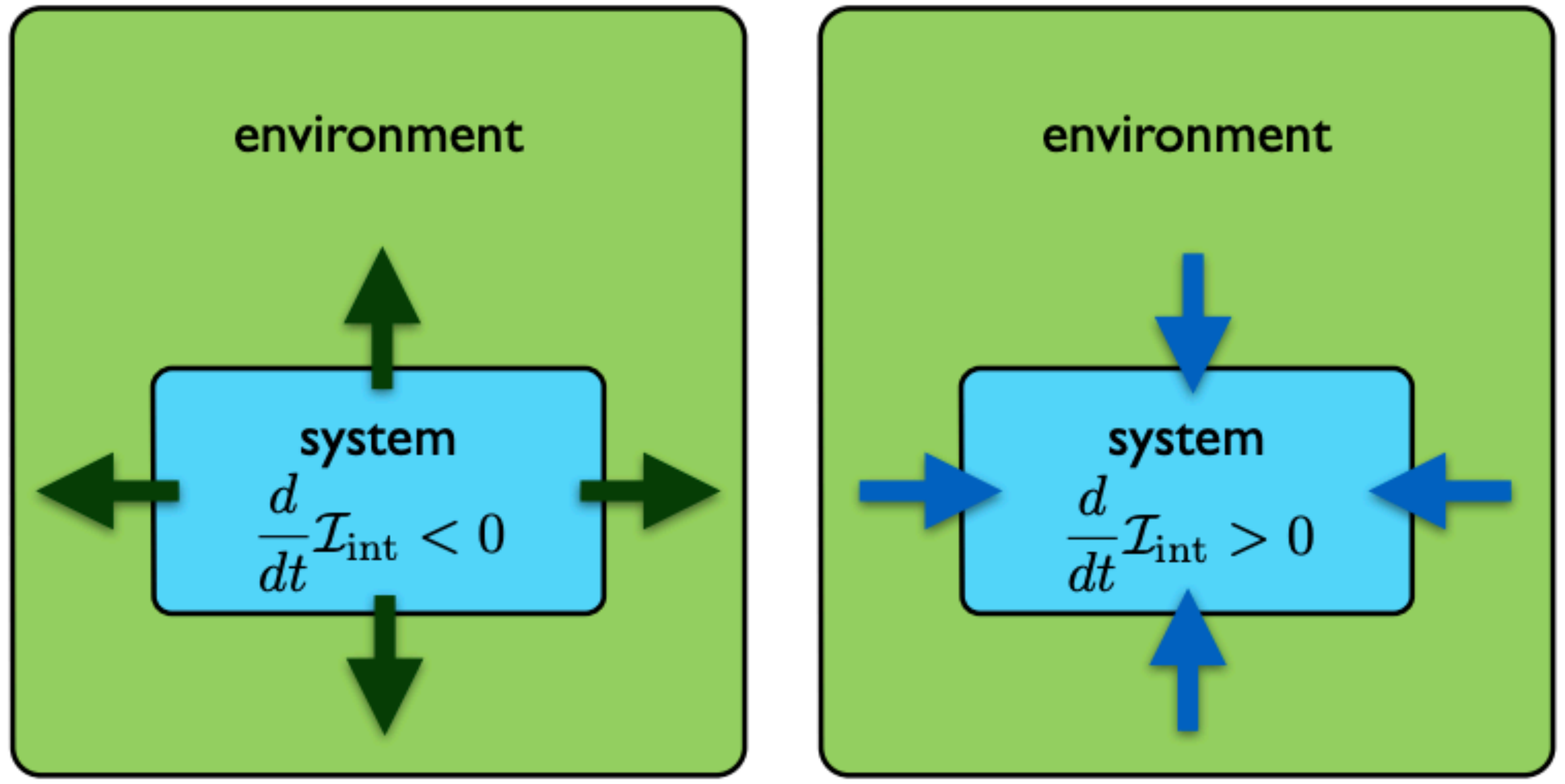
Distinguishability

Quantum process Φ_t non-Markovian if there is some pair of initial states $\{\rho_S^1(0), \rho_S^2(0)\}$ such that

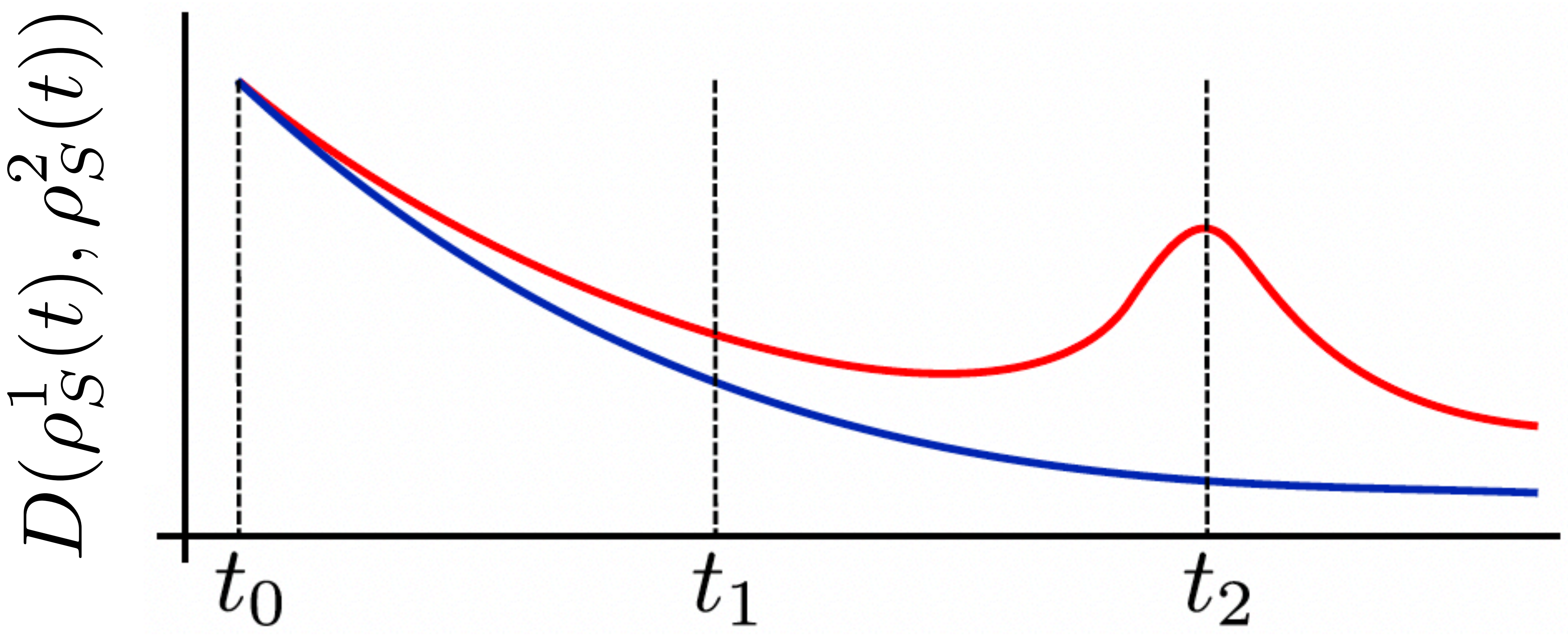
$$D(\rho_S^1(t_2), \rho_S^2(t_2)) \geq D(\rho_S^1(t_1), \rho_S^2(t_1))$$

$$t_2 > t_1$$

$$\mathcal{I}_{\text{int}} = D(\rho_S^1(t), \rho_S^2(t))$$



H.-P. Breuer et al PRL 103, 210401 (2009)



Divisibility $\Rightarrow$ / $\not\Leftarrow$ Decreasing state distinguishability (DSD)

A. Rivas et al RPP 77, 94001 (2014)

Outline

Stellenbosch UNIVERSITY IYUNIVESITHI UNIVERSITEIT

- Motivations.
- Open quantum systems - Markov approximation.
- Markovianity in classical stochastic processes.
- Quantum non-Markovianity - divisibility, distinguishability.
- Example: Spontaneous emission of two-level system.
- Collision models.

Example

Two-level atom + vacuum reservoir:

$$H = \omega_0 \sigma_+ \sigma_- + \sum_k \omega_k a_k^\dagger a_k + \sum_k (g_k \sigma_+ a_k + g_k^* \sigma_- a_k^\dagger) \qquad [a_k, a_{k'}^\dagger] = \delta_{kk'}$$

Example

Two-level atom + vacuum reservoir:

$$H = \omega_0 \sigma_+ \sigma_- + \sum_k \omega_k a_k^\dagger a_k + \sum_k (g_k \sigma_+ a_k + g_k^* \sigma_- a_k^\dagger) \qquad [a_k, a_{k'}^\dagger] = \delta_{kk'}$$

$$\frac{d}{dt}\rho_S(t) = \mathcal{L}_t \rho_S(t)$$

$$\mathcal{L}_t \rho_S(t) = -i\frac{S(t)}{2}[\sigma_+\sigma_-, \rho_S(t)] + \gamma(t)\left(\sigma_-\rho_S(t)\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho_S(t)\}\right)$$

see Marco's lectures

$$S(t) = -2\mathrm{Im}\left(\frac{\dot{c}_e(t)}{c_e(t)}\right) \qquad \gamma(t) = -2\mathrm{Re}\left(\frac{\dot{c}_e(t)}{c_e(t)}\right)$$

Lamb shift — Decay rate

Example

Two-level atom + vacuum reservoir:

$$\rho_S(t) = \begin{pmatrix} |c_e(t)|^2 & c_g^* c_e(t) \\ c_g c_e^*(t) & 1 - |c_e(t)|^2 \end{pmatrix}$$

$$f(\tau) = \mathrm{Tr}_E[e^{iH_E\tau} B e^{-iH_E\tau} B|\vec{0}\rangle\langle\vec{0}|]e^{i\omega_0\tau}$$

$$= \int_{-\infty}^{\infty} d\omega\, J(\omega) e^{-i(\omega-\omega_0)\tau}$$

➡ $$\frac{d}{dt}c_e(t) = -\int_0^t ds\, f(t-s) c_e(s)$$

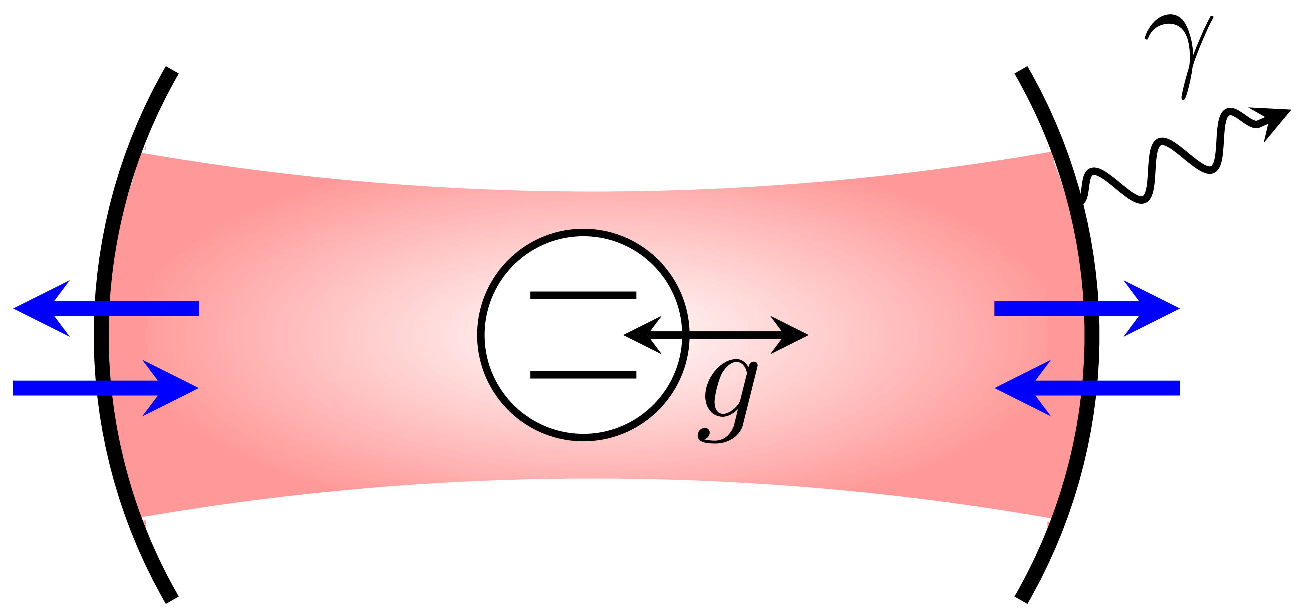
Example

Two-level atom + vacuum reservoir:

$$\rho_S(t) = \begin{pmatrix} |c_e(t)|^2 & c_g^* c_e(t) \\ c_g c_e^*(t) & 1 - |c_e(t)|^2 \end{pmatrix}$$

$$f(\tau) = \mathrm{Tr}_E[e^{iH_E\tau} B e^{-iH_E\tau} B|\vec{0}\rangle\langle\vec{0}|]e^{i\omega_0\tau}$$
$$= \int_{-\infty}^{\infty} d\omega\, J(\omega) e^{-i(\omega-\omega_0)\tau}$$

$$\frac{d}{dt}c_e(t) = -\int_0^t ds\, f(t-s)c_e(s)$$



Lorenztian spectral density:

$$J(\omega) = \frac{g^2}{\pi}\frac{\gamma/2}{(\omega-\omega_0)^2 - (\gamma/2)^2}$$

$$c_e(t) = G(t)c_e(0) = e^{-\gamma t/4}\left[\cosh(\Omega t) + \frac{\gamma}{4\Omega}\sinh(\Omega t)\right]$$

$$\Omega = \frac{1}{2}\sqrt{(\gamma/2)^2 - 4g^2}$$

Example: divisibility

Stellenbosch UNIVERSITY IYUNIVESITHI UNIVERSITEIT

Two-level atom + vacuum reservoir:

$$\mathcal{L}_t\rho_S(t) = -i\frac{S(t)}{2}[\sigma_+\sigma_-, \rho_S(t)] + \gamma(t)\left(\sigma_-\rho_S(t)\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho_S(t)\}\right)$$

$$g(t) = \lim_{\epsilon\to 0^+}\frac{\|\mathcal{I} + (\mathcal{L}_t\otimes\mathcal{I}_A^{(2)})\varepsilon\|_1 - 1}{\varepsilon} \quad \Rightarrow \quad \begin{cases} 0 & \gamma(t)\geq 0 \\ |\gamma(t)| & \gamma(t) < 0 \end{cases}$$

Example: divisibility

Stellenbosch UNIVERSITY IYUNIVESITHI UNIVERSITEIT

Two-level atom + vacuum reservoir:

$$\mathcal{L}_t \rho_S(t) = -i\frac{S(t)}{2}[\sigma_+\sigma_-, \rho_S(t)] + \gamma(t)\left(\sigma_-\rho_S(t)\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho_S(t)\}\right)$$

$$g(t) = \lim_{\epsilon\to 0^+} \frac{\|\mathcal{I} + (\mathcal{L}_t \otimes \mathcal{I}_A^{(2)})\varepsilon\|_1 - 1}{\varepsilon} \Rightarrow \begin{cases} 0 & \gamma(t) \geq 0 \\ |\gamma(t)| & \gamma(t) < 0 \end{cases}$$

Dynamics non-Markovian for $\gamma(t) < 0$

Example: divisibility

Decay rate:

$$\gamma(t) = -2\frac{\frac{d}{dt}|G(t)|}{|G(t)|}$$

$$\gamma(t) = \frac{4g^2\sinh(\Omega t)}{2\Omega\cosh(\Omega t) + \frac{\gamma}{2}\sinh(\Omega t)}$$

Non-Markovianity (divisibility):

$$\gamma(t) < 0 \Longleftrightarrow \frac{d}{dt}|G(t)| > 0$$

Example: divisibility

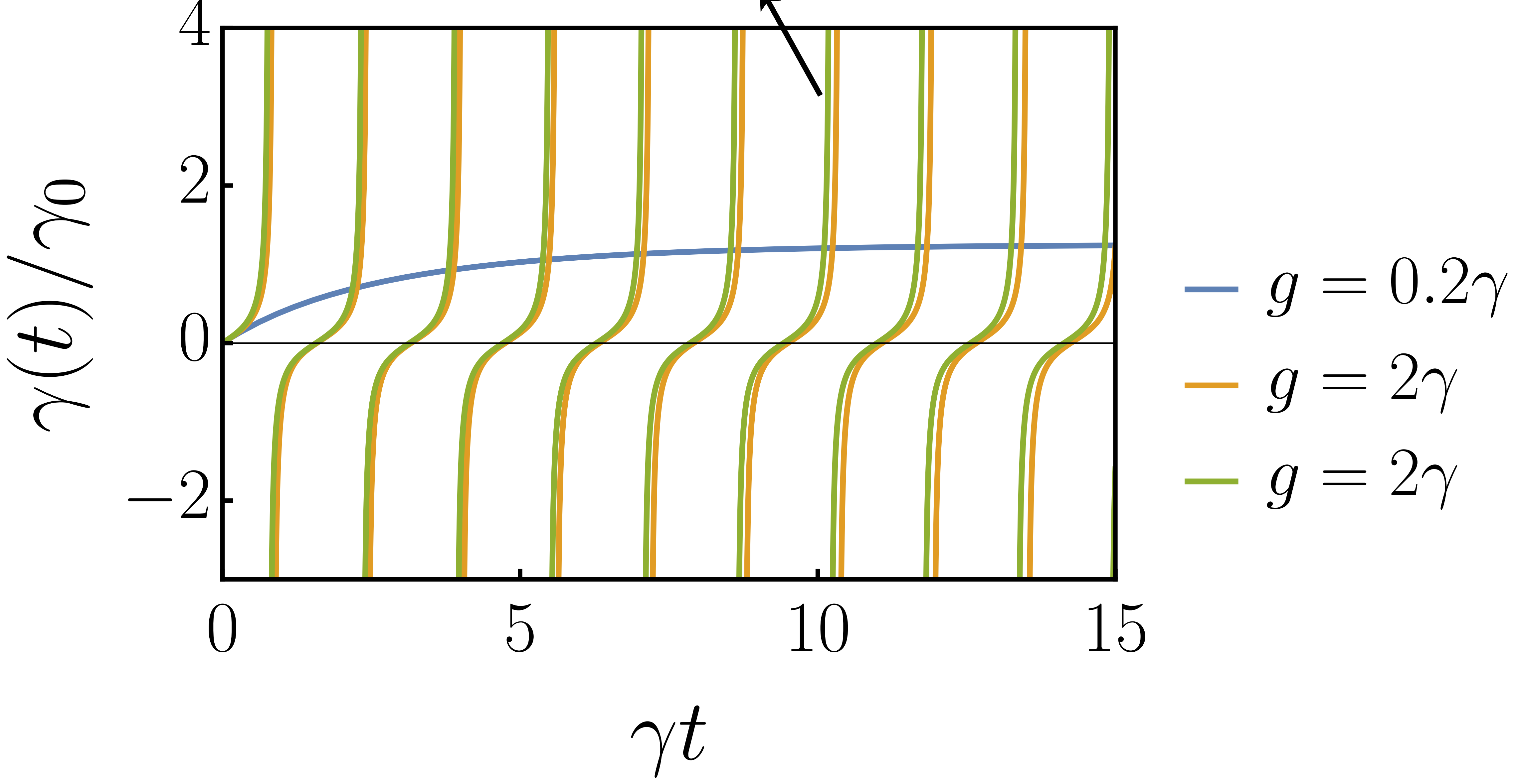
Decay rate:

$$\gamma(t) = -2\frac{\frac{d}{dt}|G(t)|}{|G(t)|}$$

Non-Markovianity (divisibility):

$$\gamma(t) < 0 \Longleftrightarrow \frac{d}{dt}|G(t)| > 0$$

$$\gamma(t) = \frac{4g^2\sinh(\Omega t)}{2\Omega\cosh(\Omega t) + \frac{\gamma}{2}\sinh(\Omega t)}$$



$\gamma > 4g$ Markovian

$4g > \gamma$ non-Markovian $\gamma(t) = \dfrac{4g^2}{2\tilde{\Omega}\cot(\tilde{\Omega}t) + \frac{\gamma}{2}}$

$g \gg \gamma$ $\gamma(t) \approx 2g\tan(gt)$

$$\Omega = i\tilde{\Omega} = \frac{i}{2}\sqrt{4g^2 - (\gamma/2)^2} \qquad \gamma_0 = \frac{4g^2}{\gamma}$$

Example: DSD

Two-level atom + vacuum reservoir:

$$\rho_S^{1,2}(t) = \begin{pmatrix} |G(t)|^2 \rho_{ee}^{1,2} & G(t)\rho_{eg}^{1,2} \\ G^*(t) {\rho^*}_{eg}^{1,2} & 1 - |G(t)|^2 \rho_{ee}^{1,2} \end{pmatrix}$$

$$\frac{1}{2}\|\Delta\rho_S(t)\|_1 = \frac{1}{2}|\lambda_-| + \frac{1}{2}|\lambda_+|$$

$\lambda_\pm$ eigenvalues of $\Delta\rho_S^{1,2} = \rho_S^1 - \rho_S^2$

Example: DSD

Stellenbosch UNIVERSITY IYUNIVESITHI UNIVERSITEIT

Two-level atom + vacuum reservoir:

$$\rho_S^{1,2}(t) = \begin{pmatrix} |G(t)|^2 \rho_{ee}^{1,2} & G(t)\rho_{eg}^{1,2} \\ G^*(t){\rho^*}_{eg}^{1,2} & 1 - |G(t)|^2 \rho_{ee}^{1,2} \end{pmatrix}$$

$$\frac{1}{2}\|\Delta\rho_S(t)\|_1 = \frac{1}{2}|\lambda_-| + \frac{1}{2}|\lambda_+|$$

$\lambda_\pm$ eigenvalues of $\Delta\rho_S^{1,2} = \rho_S^1 - \rho_S^2$

$$\frac{1}{2}\|\Delta\rho_S(t)\|_1 = |G(t)|\sqrt{|G(t)|^2\Delta\rho_{ee}^2 - |\Delta\rho_{eg}|^2}$$

$$\Rightarrow \quad \frac{d}{dt}D(\rho_S^1(t), \rho_S^2(t)) = \frac{\frac{d}{dt}|G(t)|(|G(t)|^2\Delta\rho_{ee}^2 - |\Delta\rho_{eg}|^2)}{\sqrt{|G(t)|^2\Delta\rho_{ee}^2 - |\Delta\rho_{eg}|^2}}$$

Example: DSD

Two-level atom + vacuum reservoir:

$$\rho_S^{1,2}(t) = \begin{pmatrix} |G(t)|^2 \rho_{ee}^{1,2} & G(t)\rho_{eg}^{1,2} \\ G^*(t)\rho^{*\,1,2}_{eg} & 1 - |G(t)|^2 \rho_{ee}^{1,2} \end{pmatrix}$$

$$\frac{1}{2}\|\Delta\rho_S(t)\|_1 = \frac{1}{2}|\lambda_-| + \frac{1}{2}|\lambda_+|$$

$\lambda_\pm$ eigenvalues of $\Delta\rho_S^{1,2} = \rho_S^1 - \rho_S^2$

$$\frac{1}{2}\|\Delta\rho_S(t)\|_1 = |G(t)|\sqrt{|G(t)|^2 \Delta\rho_{ee}^2 - |\Delta\rho_{eg}|^2}$$

$$\Rightarrow \quad \frac{d}{dt} D(\rho_S^1(t), \rho_S^2(t)) = \frac{\frac{d}{dt}|G(t)|(|G(t)|^2 \Delta\rho_{ee}^2 - |\Delta\rho_{eg}|^2)}{\sqrt{|G(t)|^2 \Delta\rho_{ee}^2 - |\Delta\rho_{eg}|^2}}$$

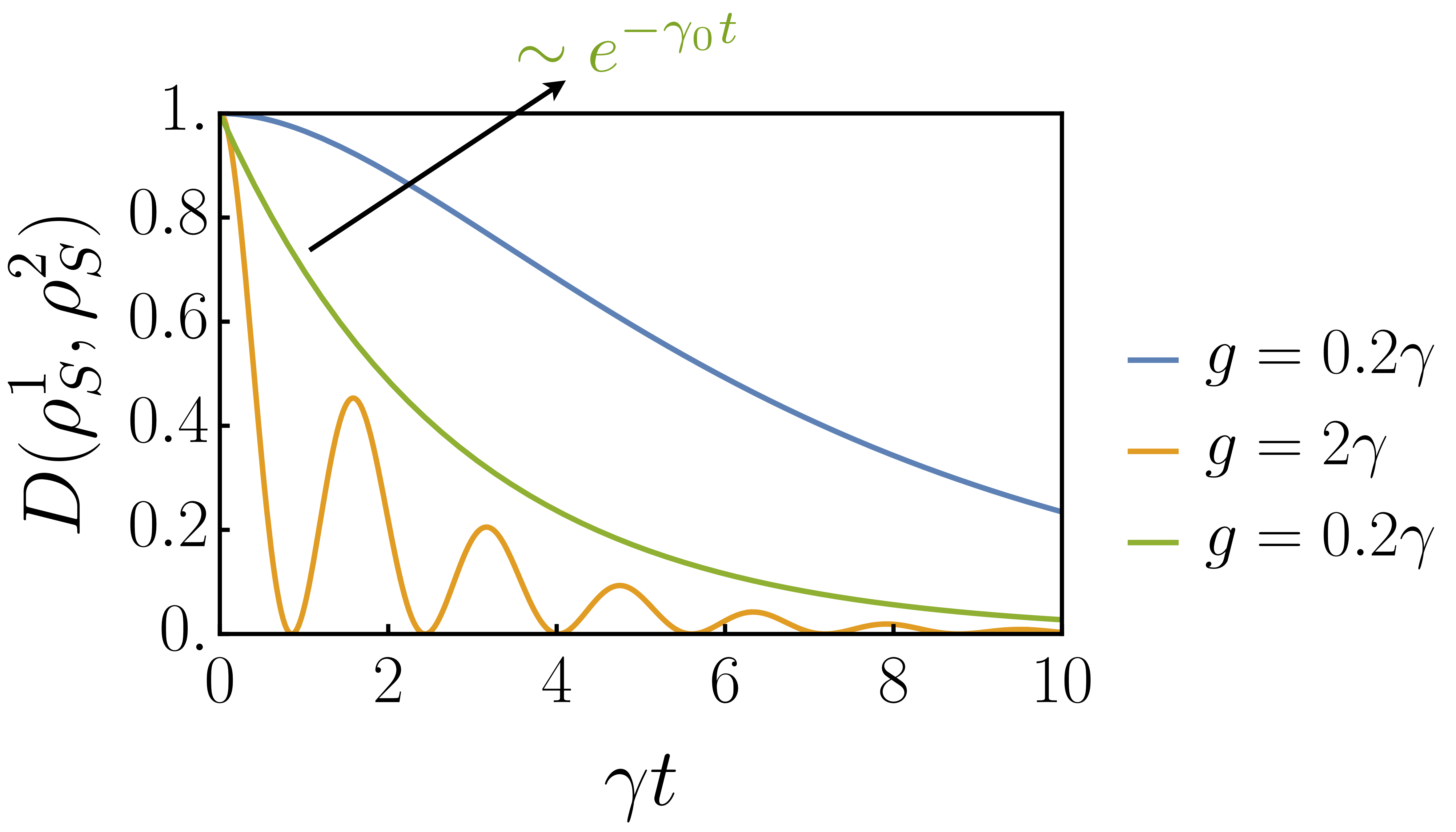
Dynamics non-Markovian for $\frac{d}{dt}|G(t)| > 0$ $\Leftrightarrow$ $\gamma(t) < 0$

Example: DSD

Trace distance:

$$\frac{1}{2}\|\Delta\rho_S(t)\|_1 = |G(t)|\sqrt{|G(t)|^2\Delta\rho_{ee}^2 - |\Delta\rho_{eg}|^2}$$

$$\frac{d}{dt}D(\rho_S^1, \rho_S^2) > 0 \iff \gamma(t) < 0$$



$1 \rightarrow$ Initially excited $\Delta\rho_{ee} = 1$ $\Delta\rho_{eg} = 0$

$2 \rightarrow$ Ground state

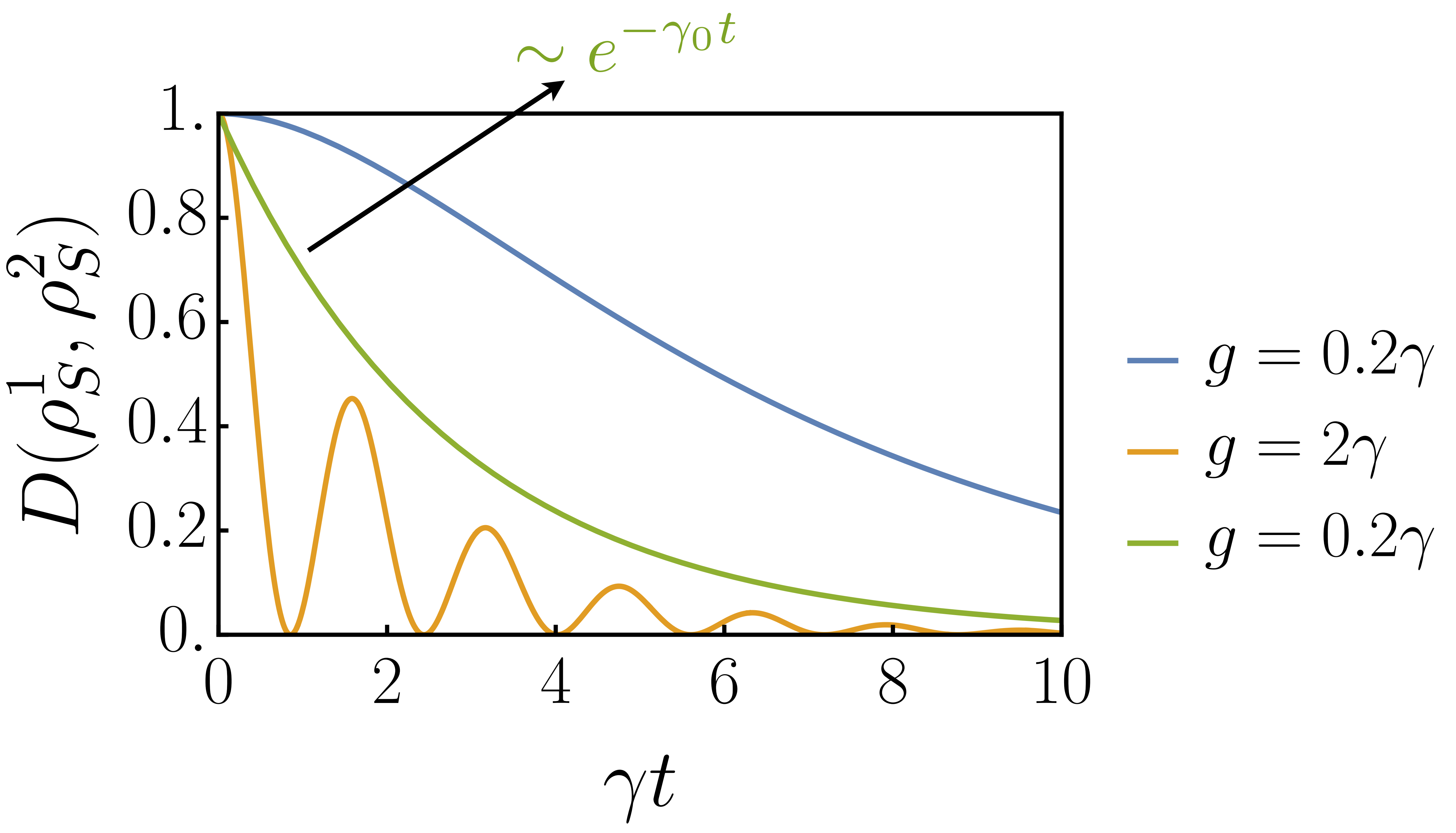
Example: DSD

Trace distance:

$$\frac{1}{2}\|\Delta\rho_S(t)\|_1 = |G(t)|\sqrt{|G(t)|^2\Delta\rho_{ee}^2 - |\Delta\rho_{eg}|^2}$$

$$\frac{d}{dt}D(\rho_S^1, \rho_S^2) > 0 \quad \Longleftrightarrow \quad \gamma(t) < 0$$

Divisibility $\Longleftrightarrow$ DSD



$1 \rightarrow$ Initially excited $\quad \Delta\rho_{ee} = 1 \quad \Delta\rho_{eg} = 0$

$2 \rightarrow$ Ground state

Outline

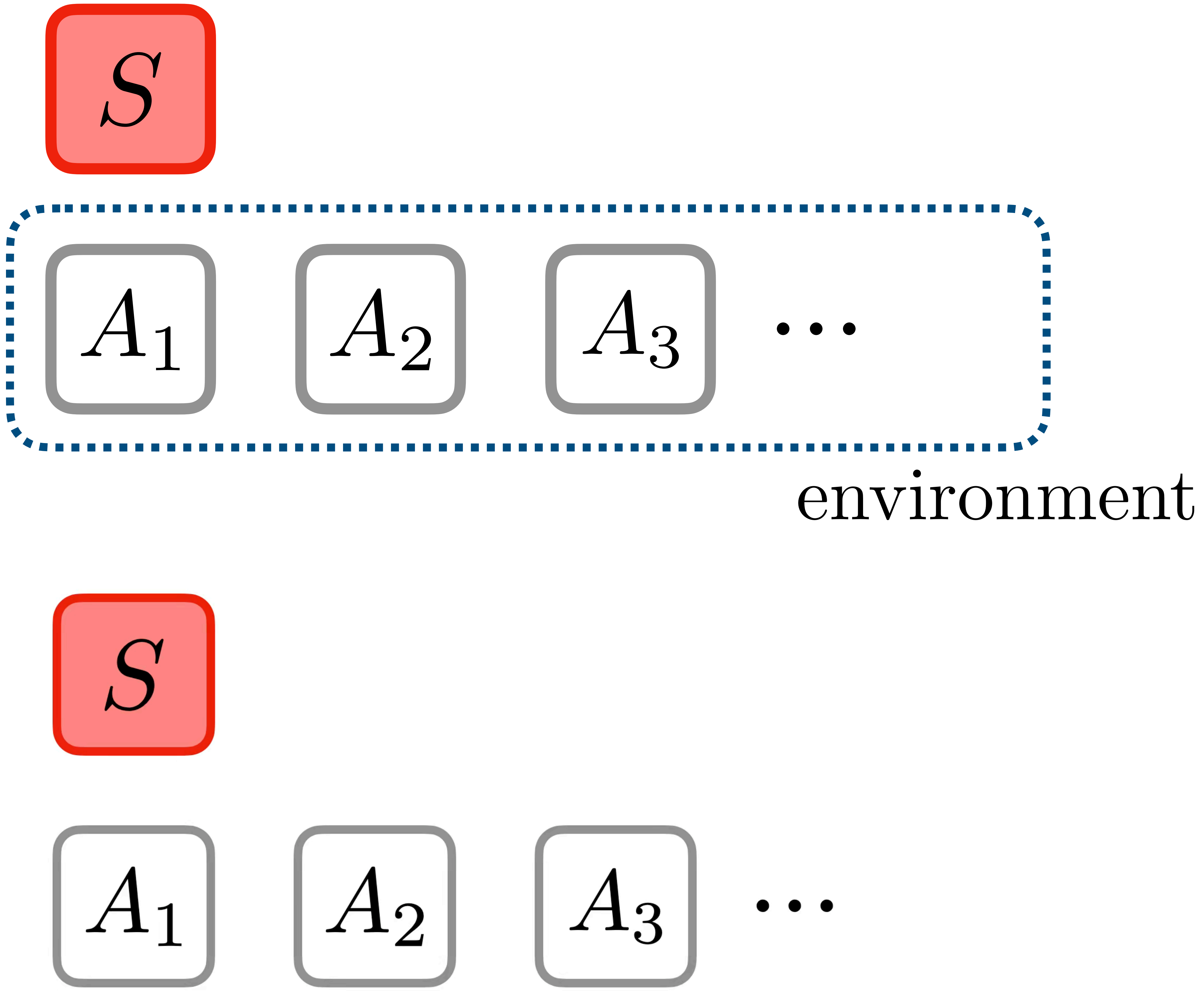
Stellenbosch UNIVERSITY IYUNIVESITHI UNIVERSITEIT

- Motivations.
- Open quantum systems - Markov approximation.
- Markovianity in classical stochastic processes.
- Quantum non-Markovianity - divisibility, distinguishability.
- Example: Spontaneous emission of two-level system.
- Collision models.

Collision models (CMs)

- Same system+environment partition, but environment now made up of discrete ancillas.
- Collisions occur in sequence.
- New ancillas added after each collision:

$$\rho \in \mathcal{S}(\mathcal{H}_S \otimes \mathcal{H}_{A_1} \otimes \ldots \otimes \mathcal{H}_{A_n})$$



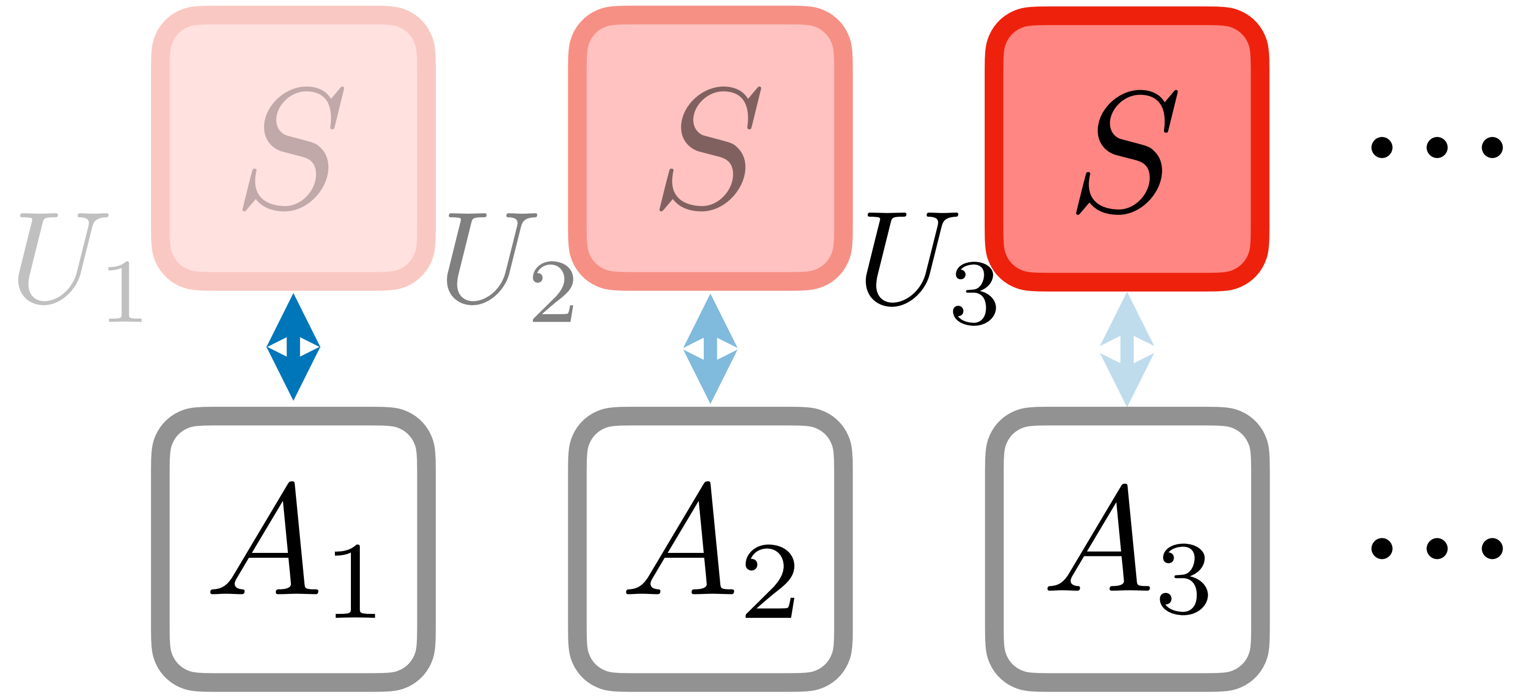
F. Ciccarello et al., PR 954, 1 (2022)

Basic collision models

- Basic CM assumptions:
  - Components initially uncorrelated:

$$\rho(0) = \rho_S(0) \otimes \rho_{A_1} \otimes \ldots \otimes \rho_{A_n}$$

  - System collides with each ancilla once:



$$\rho(n) = \mathcal{U}_n \circ \ldots \circ \mathcal{U}_1 \rho(0)$$
$$= U_n \big(U_{n-1} \ldots \big(U_1 \rho(0) U_1^\dagger\big) \ldots U_{n-1}^\dagger\big) U_n^\dagger$$

  - No ancilla-ancilla collisions.

F. Ciccarello et al., PR 954, 1 (2022)

Basic collision models

- `Time' evolution:

$$\rho(n) = U\big(U...\big(U(\rho_S(0) \otimes \sigma)U^\dagger\big)...\sigma U^\dagger\big)\sigma U^\dagger$$

*(ancillas in same initial state)

- Reduced system state:

$$\rho_S(n) = \mathrm{Tr}_{A_1...A_n}[\rho(n)]$$

$$= \mathrm{Tr}_{A_n}[U\mathrm{Tr}_{A_{n-1}}[U...\mathrm{Tr}_{A_1}[U(\rho_S(0) \otimes \sigma)U^\dagger]...\sigma U^\dagger]\sigma U^\dagger]$$

Basic collision models

Stellenbosch UNIVERSITY IYUNIVESITHI UNIVERSITEIT

- `Time' evolution:

$$\rho(n) = U\big(U\ldots\big(U(\rho_S(0)\otimes\sigma)U^\dagger\big)\ldots\sigma U^\dagger\big)\sigma U^\dagger$$

*(ancillas in same initial state)

- Reduced system state:

$$\begin{aligned}\rho_S(n) &= \mathrm{Tr}_{A_1\ldots A_n}[\rho(n)] \\ &= \mathrm{Tr}_{A_n}[U\mathrm{Tr}_{A_{n-1}}[U\ldots\mathrm{Tr}_{A_1}[U(\rho_S(0)\otimes\sigma)U^\dagger]\ldots\sigma U^\dagger]\sigma U^\dagger]\end{aligned}$$

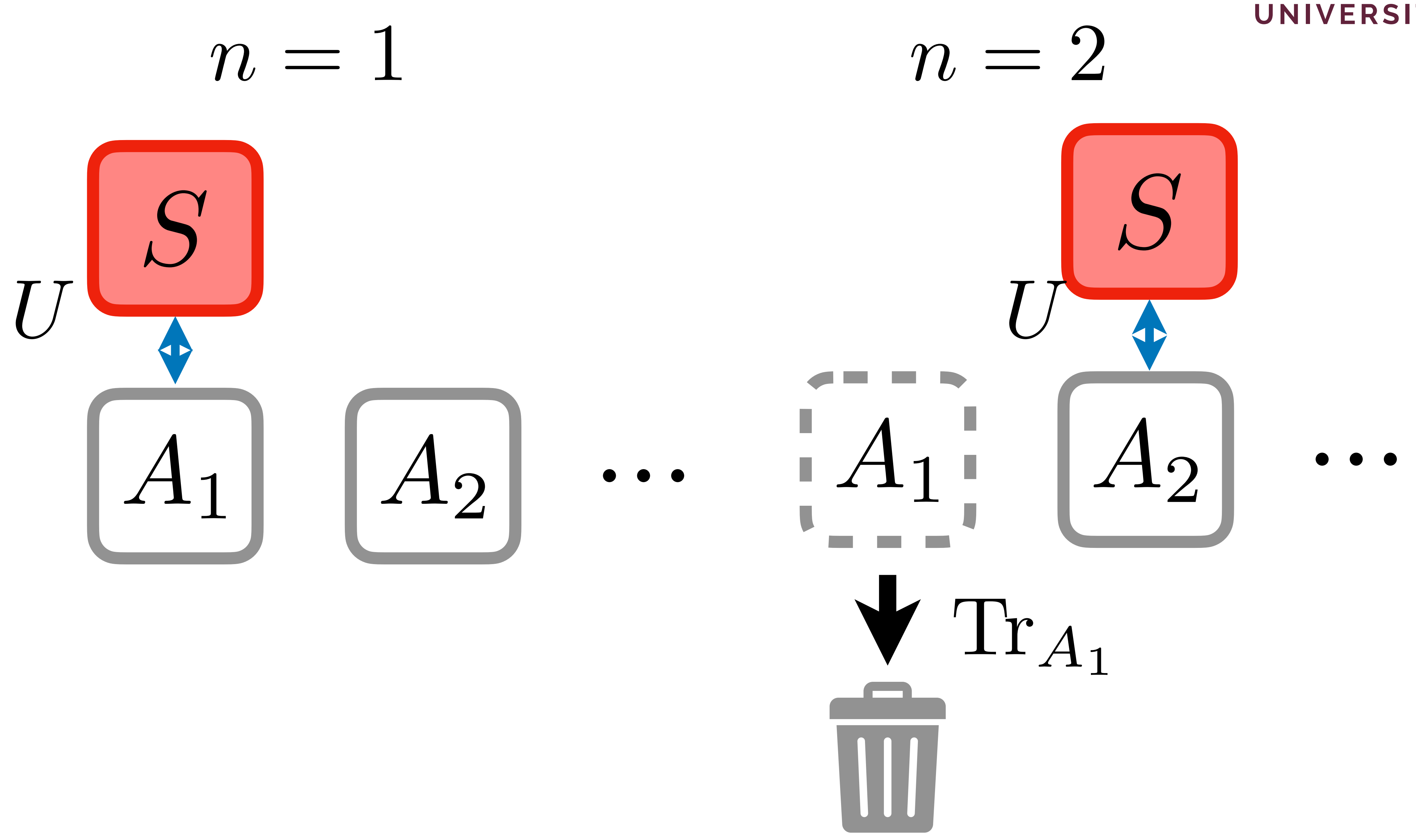
- Dynamical map $\Phi : \mathcal{S}(\mathcal{H}_S) \to \mathcal{S}(\mathcal{H}_S)$

$$\boxed{\Phi[\rho_S] = \mathrm{Tr}_A[U(\rho_S\otimes\sigma)U^\dagger]}$$

Basic collision models

- Reduced system state after n collisions:

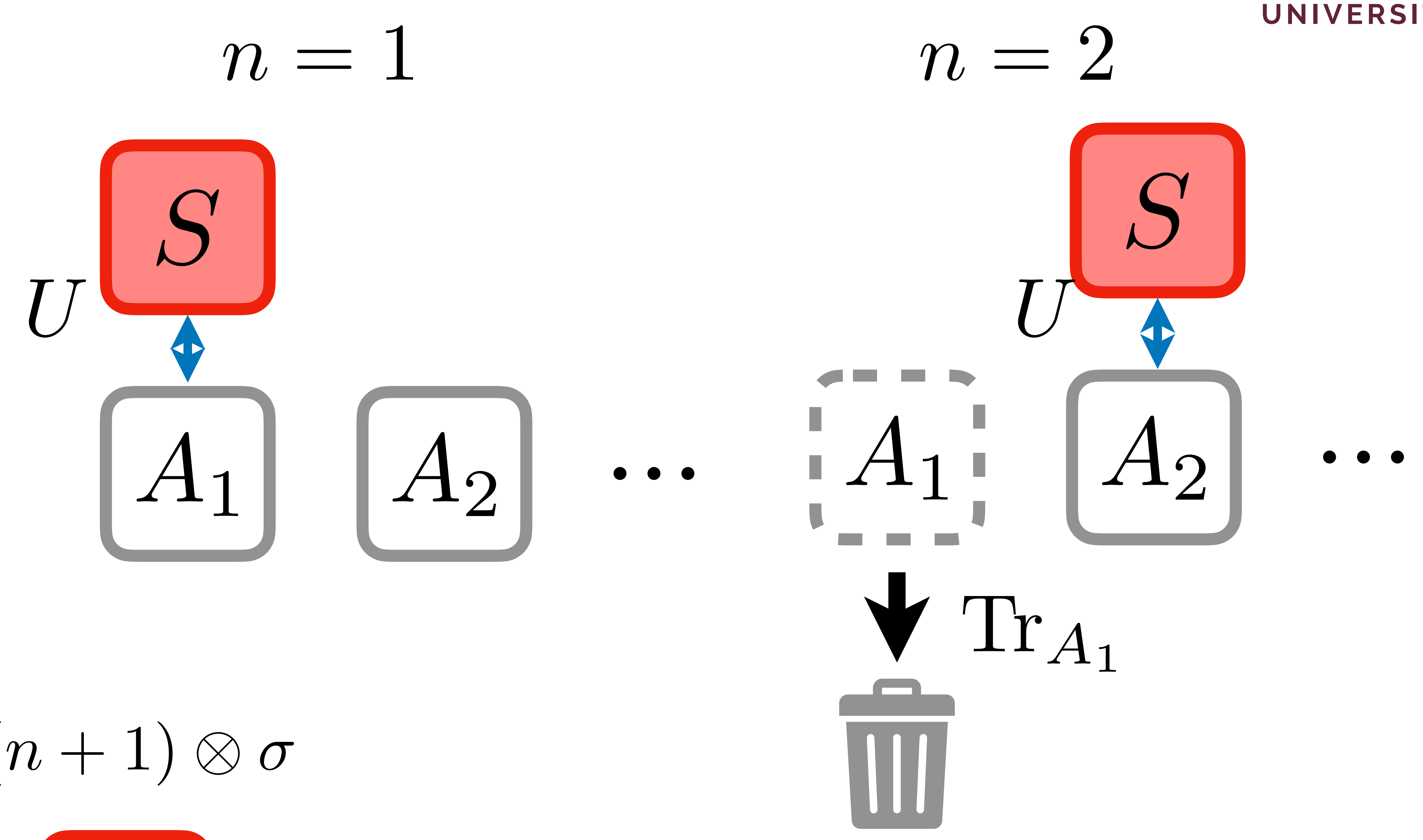
$$\rho_S(n) = \Phi \circ \Phi \ldots \circ \Phi[\rho_S(0)]$$
$$= \Phi^n[\rho_S(0)]$$



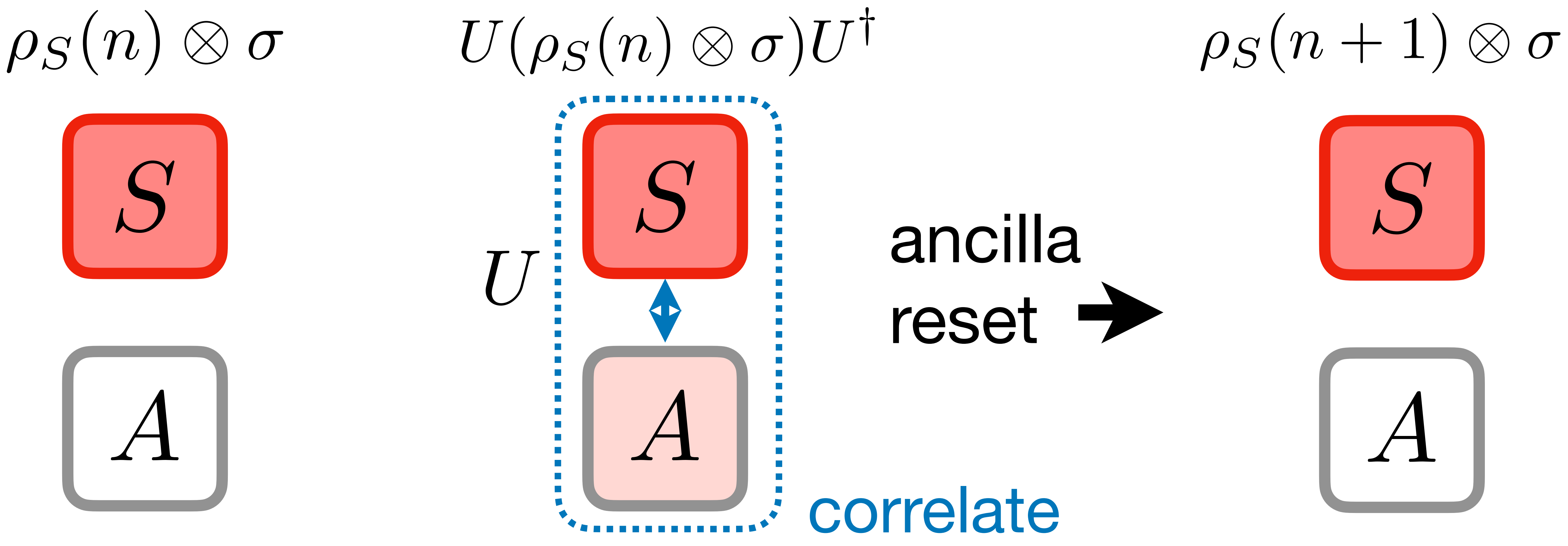
Basic collision models

- Reduced system state after n collisions:

$$\rho_S(n) = \Phi \circ \Phi \ldots \circ \Phi[\rho_S(0)]$$
$$= \Phi^n[\rho_S(0)]$$



- Setup describes Markovian dynamics:



Basic collision models

- Consider one-parameter family of CPTP maps $\{\Phi_n \,|\, n \in \mathbb{N},\ \Phi_0 = \mathbb{I}\}$:

$$\Phi_n = \Lambda_{n,m} \circ \Phi_m \quad \forall n, m$$

Basic collision models

- Consider one-parameter family of CPTP maps $\{\Phi_n \,|\, n \in \mathbb{N},\ \Phi_0 = \mathbb{I}\}$:

$$\Phi_n = \Lambda_{n,m} \circ \Phi_m \quad \forall n, m$$

- Divisibility:

$$\Phi_n = \Phi \circ \ldots \circ \Phi = \Phi^n \qquad \Lambda_{n,m} = \Phi_{n-m} \quad \Rightarrow$$ Markovian (semigroup)

$$\|C_{\Lambda_{n+1,n}}\|_1 = 1 \quad \forall n \qquad C_{\Lambda_{n,m}} = (\Lambda_{n,m} \otimes \mathcal{I}^{(d)})(|\Psi\rangle\langle\Psi|)$$

Basic collision models

- Consider one-parameter family of CPTP maps $\{\Phi_n \,|\, n \in \mathbb{N},\ \Phi_0 = \mathbb{I}\}$:

$$\Phi_n = \Lambda_{n,m} \circ \Phi_m \quad \forall n, m$$

- Divisibility:

$$\Phi_n = \Phi \circ \ldots \circ \Phi = \Phi^n \qquad \Lambda_{n,m} = \Phi_{n-m} \quad \Rightarrow$$ Markovian (semigroup)

$$\Rightarrow \quad \| C_{\Lambda_{n+1,n}} \|_1 = 1 \quad \forall n \qquad C_{\Lambda_{n,m}} = (\Lambda_{n,m} \otimes \mathcal{I}^{(d)})(|\Psi\rangle\langle\Psi|)$$

- Distinguishability:

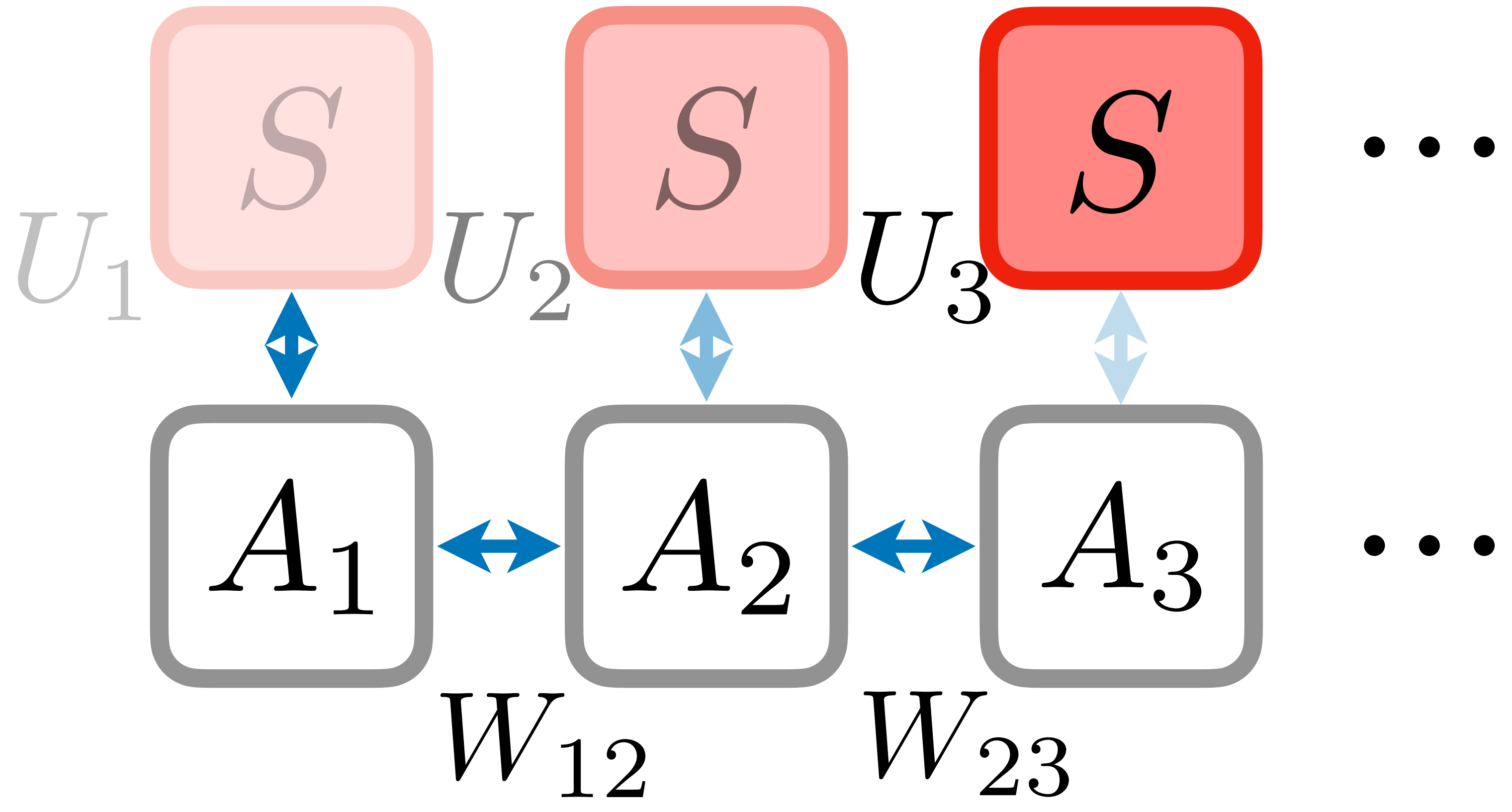
$$\Rightarrow \quad D(\rho_S^1(n+1), \rho_S^2(n+1)) \leq D(\rho_S^1(n), \rho_S^2(n))$$ ($\Lambda_{n+1,n}$ CPTP)

Non-Markovian collision models

- Ancilla-ancilla collisions:

$$\rho(n) = U'(\ldots U'(\rho_S(0) \otimes \sigma)U'^{\dagger} \ldots)\sigma U'^{\dagger}$$

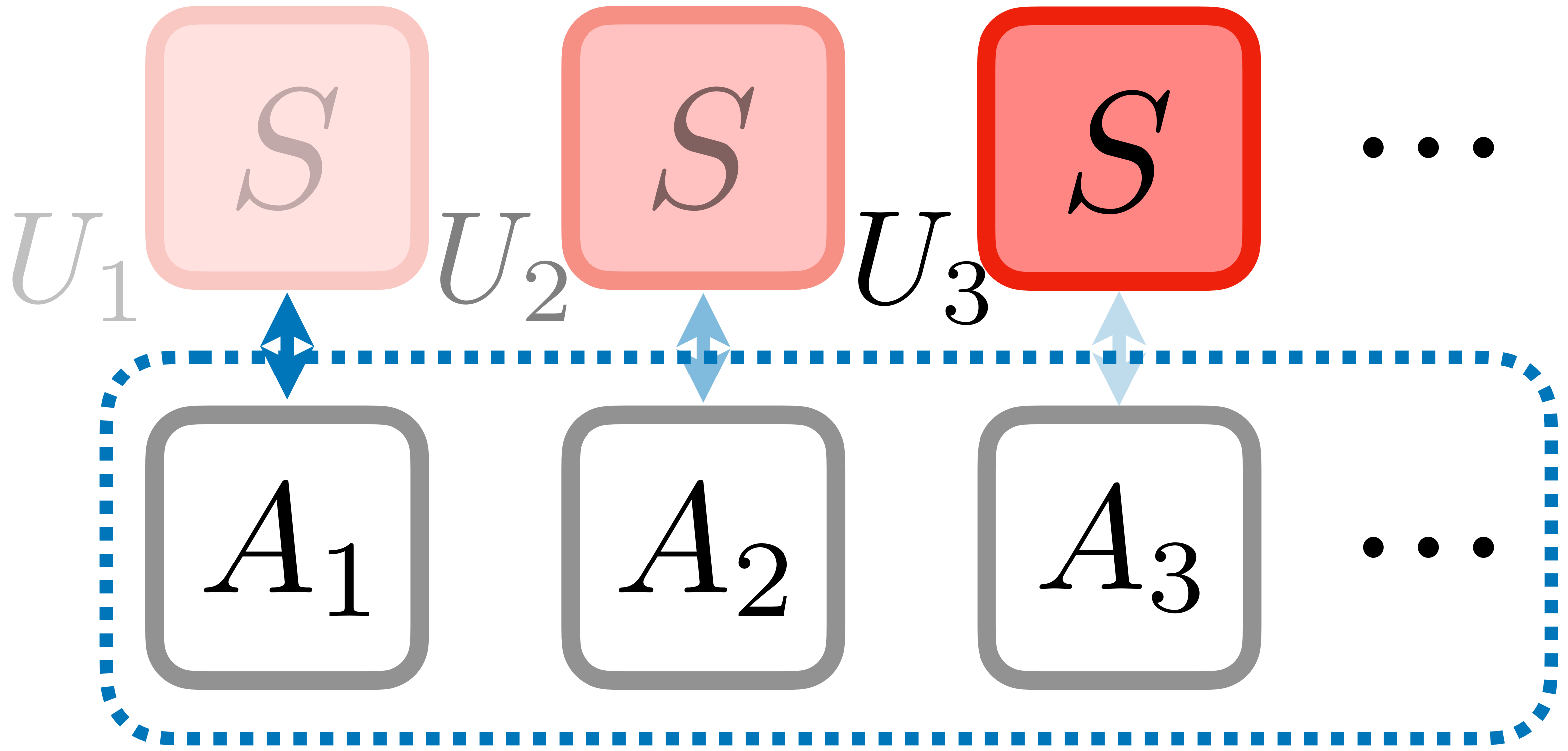
$$U' = WU$$



- Initial correlations:

$$\rho(n) = U(\ldots U(\rho_S(0) \otimes \sigma(0))U^{\dagger} \ldots)U^{\dagger}$$

$$\sigma(0) \neq \underbrace{\sigma \otimes \ldots \otimes \sigma}_{n\,\text{times}}$$

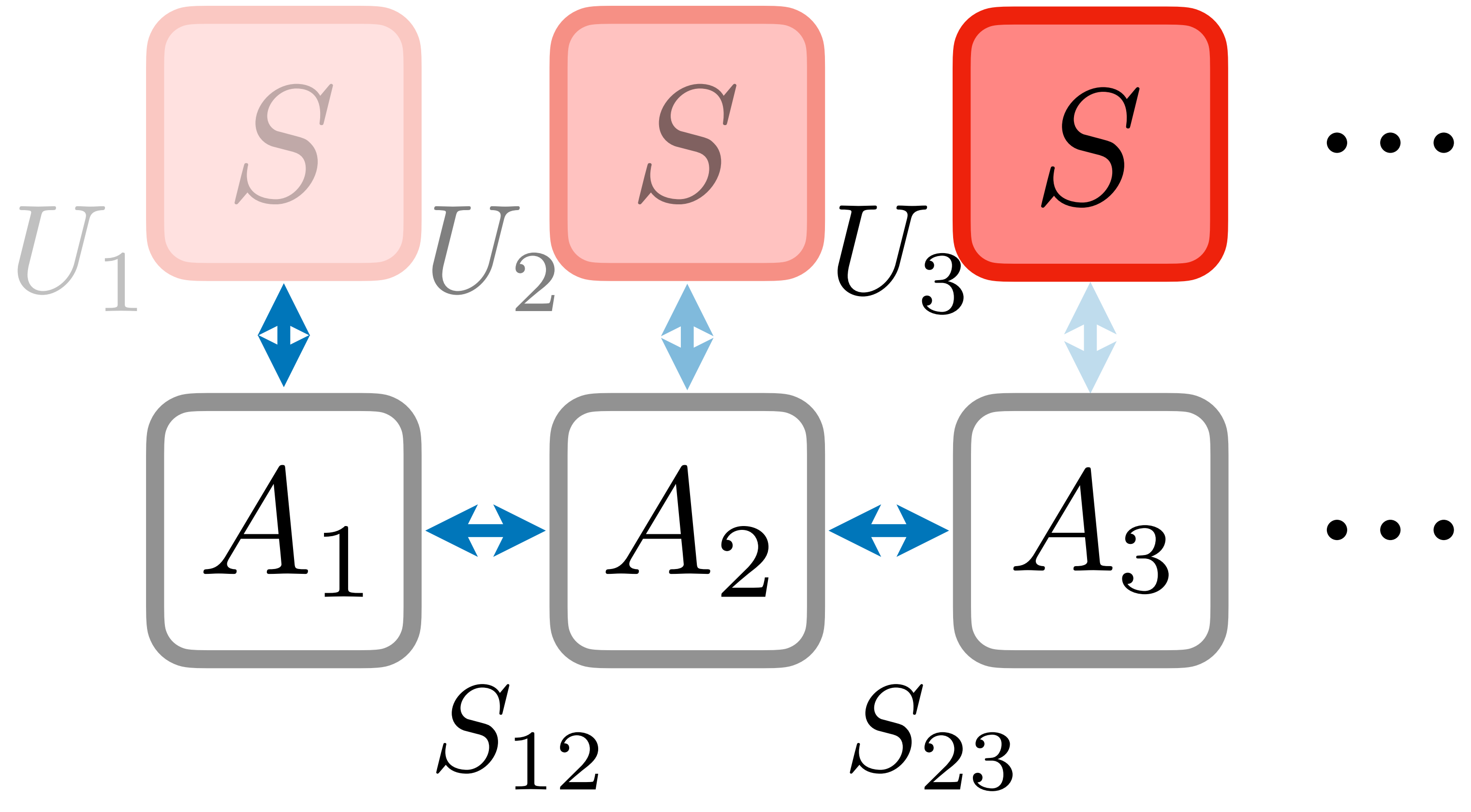


F. Ciccarello et al., PR 954, 1 (2022)

Non-Markovian collision models

- Partial swap operation:

$$\rho(n) = \mathcal{U}_n(\mathcal{S}_{n,n-1}(\cdots\mathcal{U}_1(\rho_S(0)\otimes|\vec{0}\rangle\langle\vec{0}|)\cdots))$$



Partial SWAP map (CPTP)

$$\mathcal{S}_{j+1,j}[\rho] = (1-p)\rho + pS_{j+1,j}\,\rho\, S_{j+1,j} \qquad S(\rho\otimes\sigma)S = \sigma\otimes\rho$$

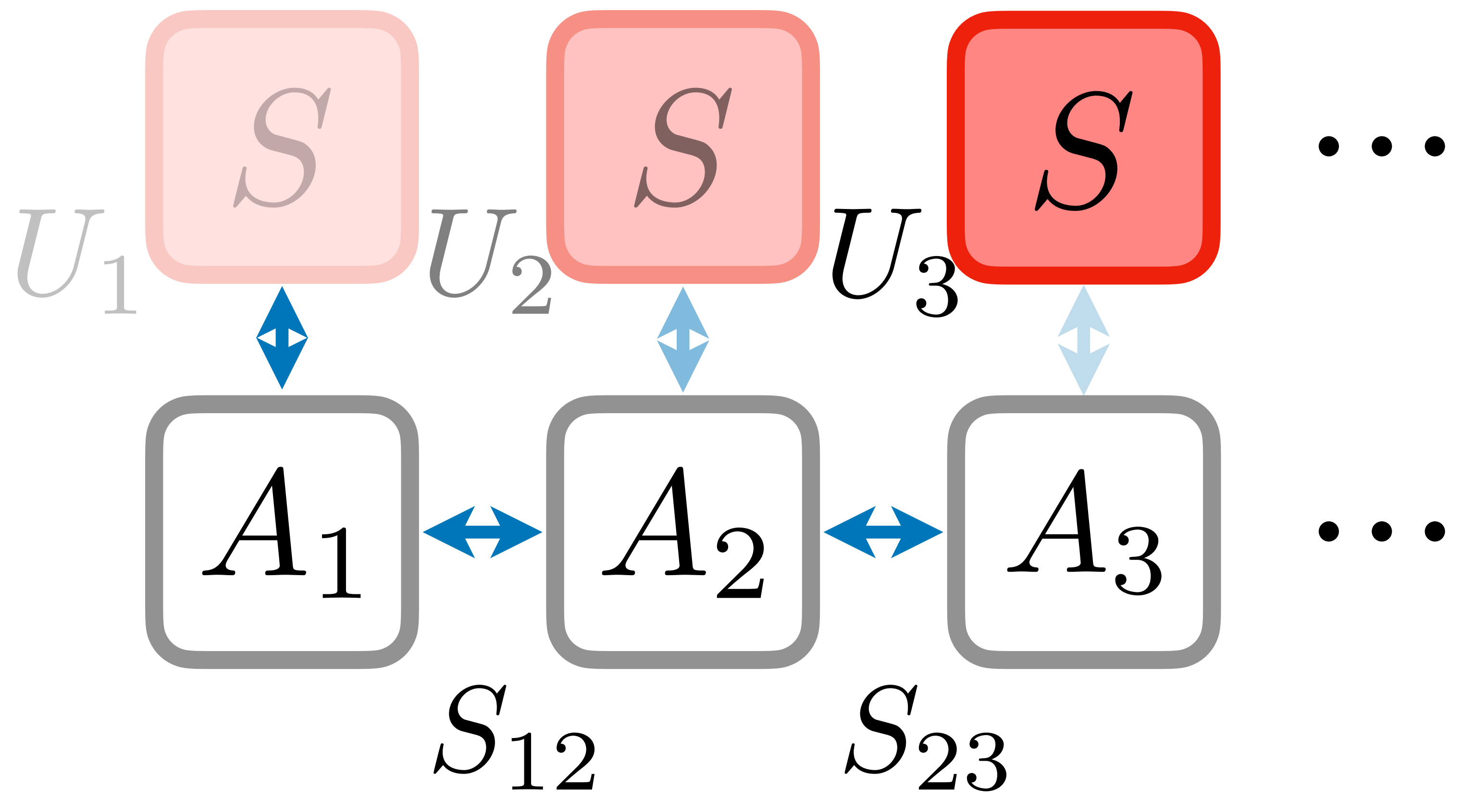
p SWAP probability

F. Ciccarello et al, PRA 87, 040103(R) (2013)

Non-Markovian collision models

- Partial swap operation:

$$\rho(n) = \mathcal{U}_n(\mathcal{S}_{n,n-1}(\cdots \mathcal{U}_1(\rho_S(0) \otimes |\vec{0}\rangle\langle\vec{0}|)\cdots))$$



Partial SWAP map (CPTP)

$$\mathcal{S}_{j+1,j}[\rho] = (1-p)\rho + pS_{j+1,j}\rho S_{j+1,j} \qquad S(\rho\otimes\sigma)S = \sigma\otimes\rho$$

p SWAP probability

$$\rho_S(n) = (1-p)\sum_{j=1}^{n-1} p^{j-1}\Phi_j[\rho_S(n-j)] + p^{n-1}\Phi_n[\rho_S(0)]$$

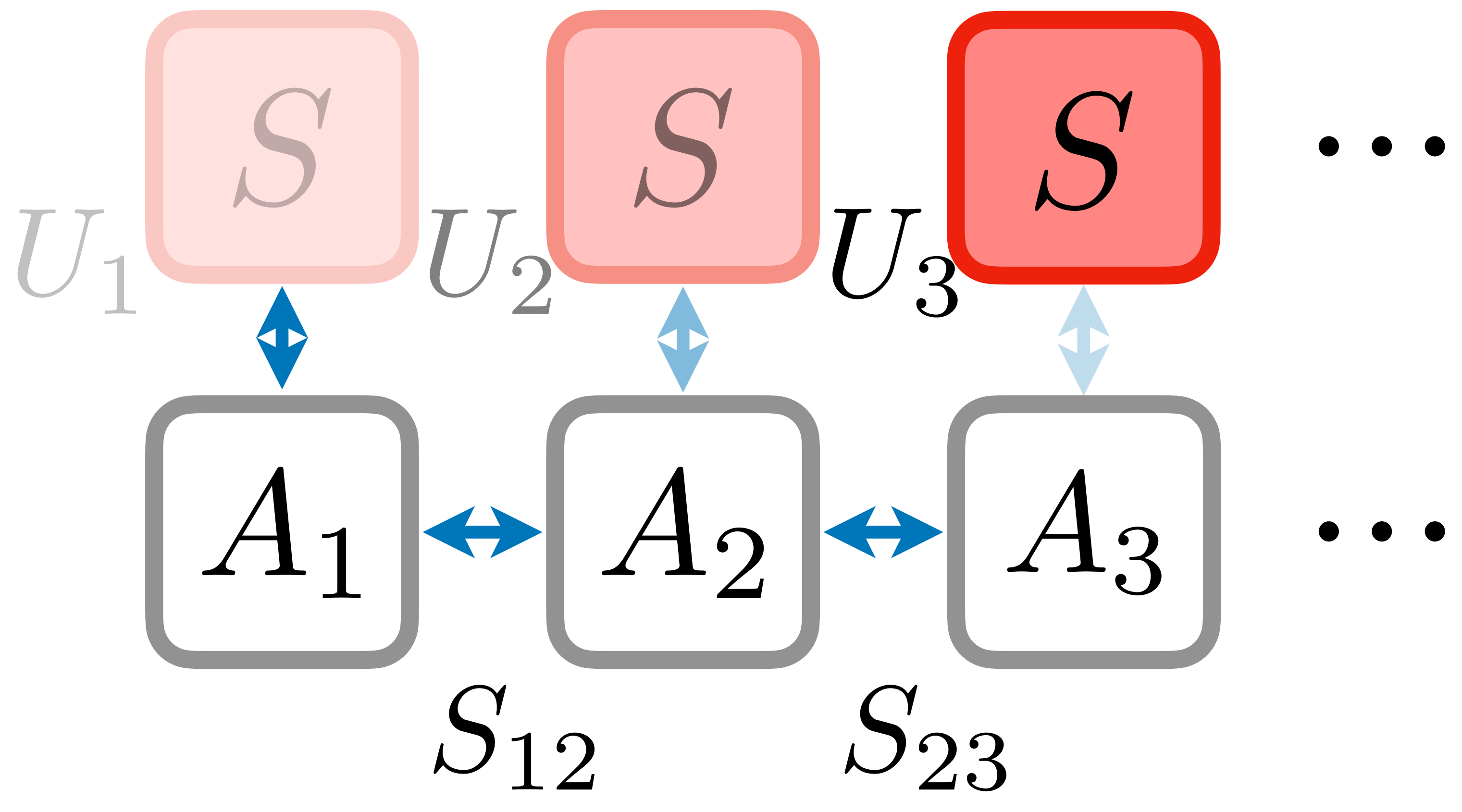
$$\Phi_j[\rho] = \mathrm{Tr}_E(\mathcal{U}_n^j[\rho\otimes|\vec{0}\rangle\langle\vec{0}|])$$

F. Ciccarello et al, PRA 87, 040103(R) (2013)

Non-Markovian collision models

- Partial swap operation:

$$\rho(n) = \mathcal{U}_n(\mathcal{S}_{n,n-1}(\cdots\mathcal{U}_1(\rho_S(0)\otimes|\vec{0}\rangle\langle\vec{0}|)\cdots))$$



Partial SWAP map (CPTP)

$$\mathcal{S}_{j+1,j}[\rho] = (1-p)\rho + pS_{j+1,j}\rho S_{j+1,j} \qquad S(\rho\otimes\sigma)S = \sigma\otimes\rho$$

$$\rho_S(n) = (1-p)\sum_{j=1}^{n-1} p^{j-1}\Phi_j[\rho_S(n-j)] + p^{n-1}\Phi_n[\rho_S(0)]$$

$$\Phi_j[\rho] = \mathrm{Tr}_E(\mathcal{U}_n^j[\rho\otimes|\vec{0}\rangle\langle\vec{0}|])$$

p SWAP probability

$p=0$ Markovian

$$\rho_S(n) = \Phi_1[\rho_S(n-1)]$$

$p=1$ non-Markovian

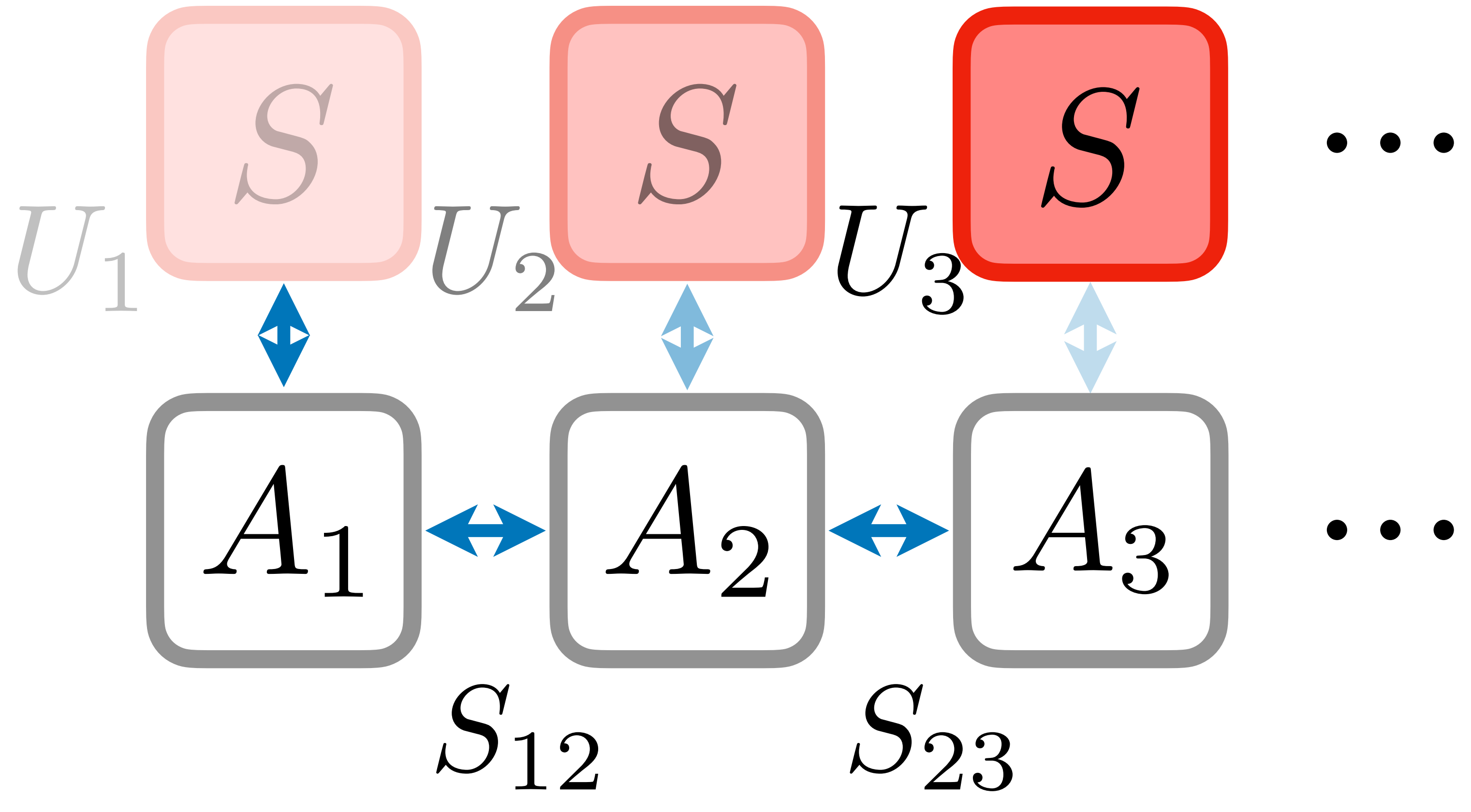
$$\rho_S(n) = \Phi_n[\rho_S(0)]$$

F. Ciccarello et al, PRA 87, 040103(R) (2013)

Non-Markovian collision models

- Partial swap operation:

$$\rho(n) = \mathcal{U}_n(\mathcal{S}_{n,n-1}(\cdots\mathcal{U}_1(\rho_S(0)\otimes|\vec{0}\rangle\langle\vec{0}|)\cdots))$$



Partial SWAP map (CPTP)

$$\mathcal{U}_j\rho = e^{-iH_{I_j}\tau}\rho e^{iH_{I_j}\tau} \qquad p = e^{-\gamma\tau}$$

Continuous `time' limit: $t = n\tau \qquad \tau\to 0 \quad n\to\infty$

γ^{-1} memory time

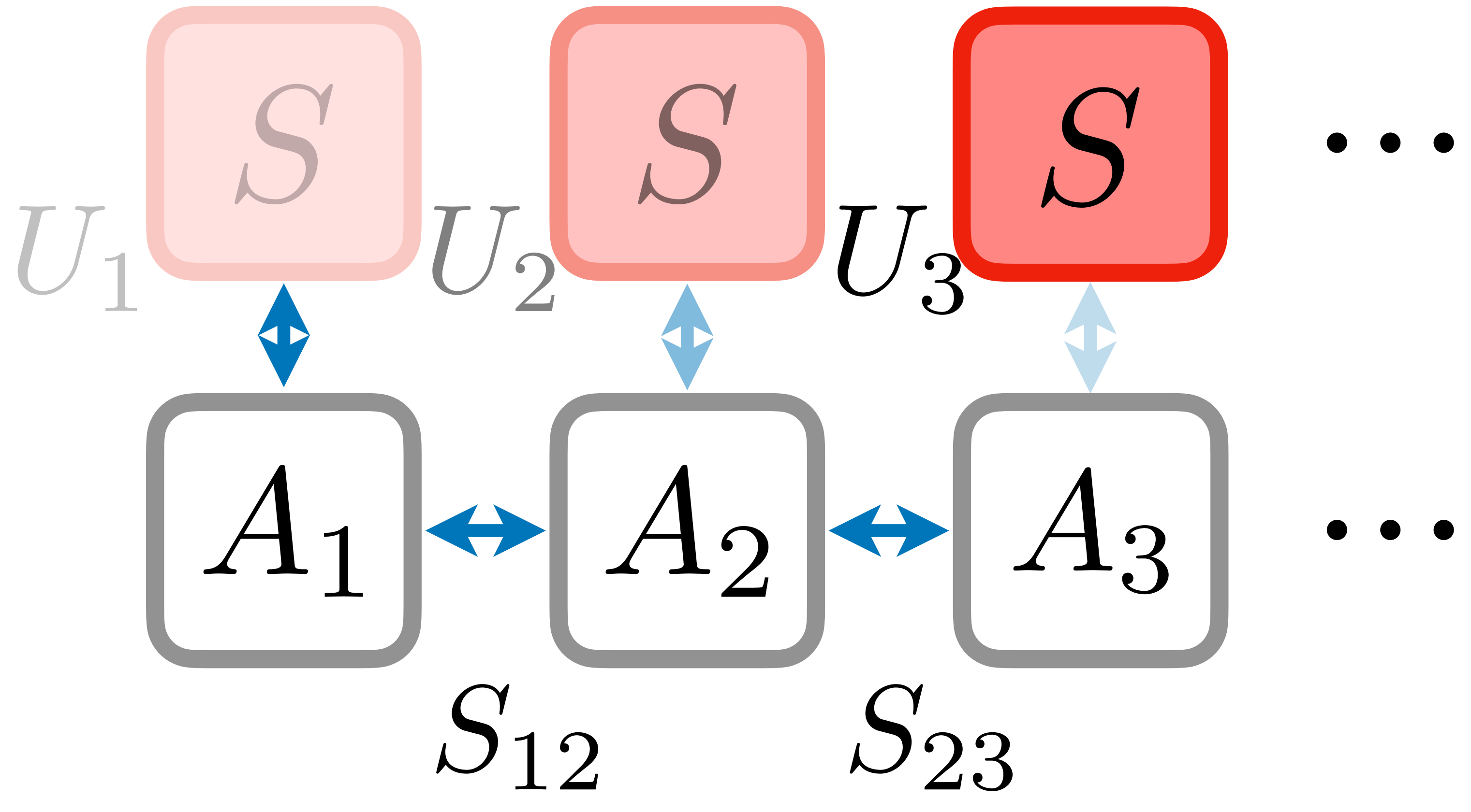
$$\dot{\rho}_S(t) = \gamma\int_0^t ds\, e^{-\gamma s}\Phi_s\dot{\rho}_S(t-s) + e^{-\gamma t}\dot{\Phi}_t[\rho_S(0)]$$

F. Ciccarello et al, PRA 87, 040103(R) (2013)

Non-Markovian collision models

- Partial swap operation:

$$\rho(n) = \mathcal{U}_n(\mathcal{S}_{n,n-1}(\cdots \mathcal{U}_1(\rho_S(0) \otimes |\vec{0}\rangle\langle\vec{0}|)\cdots))$$



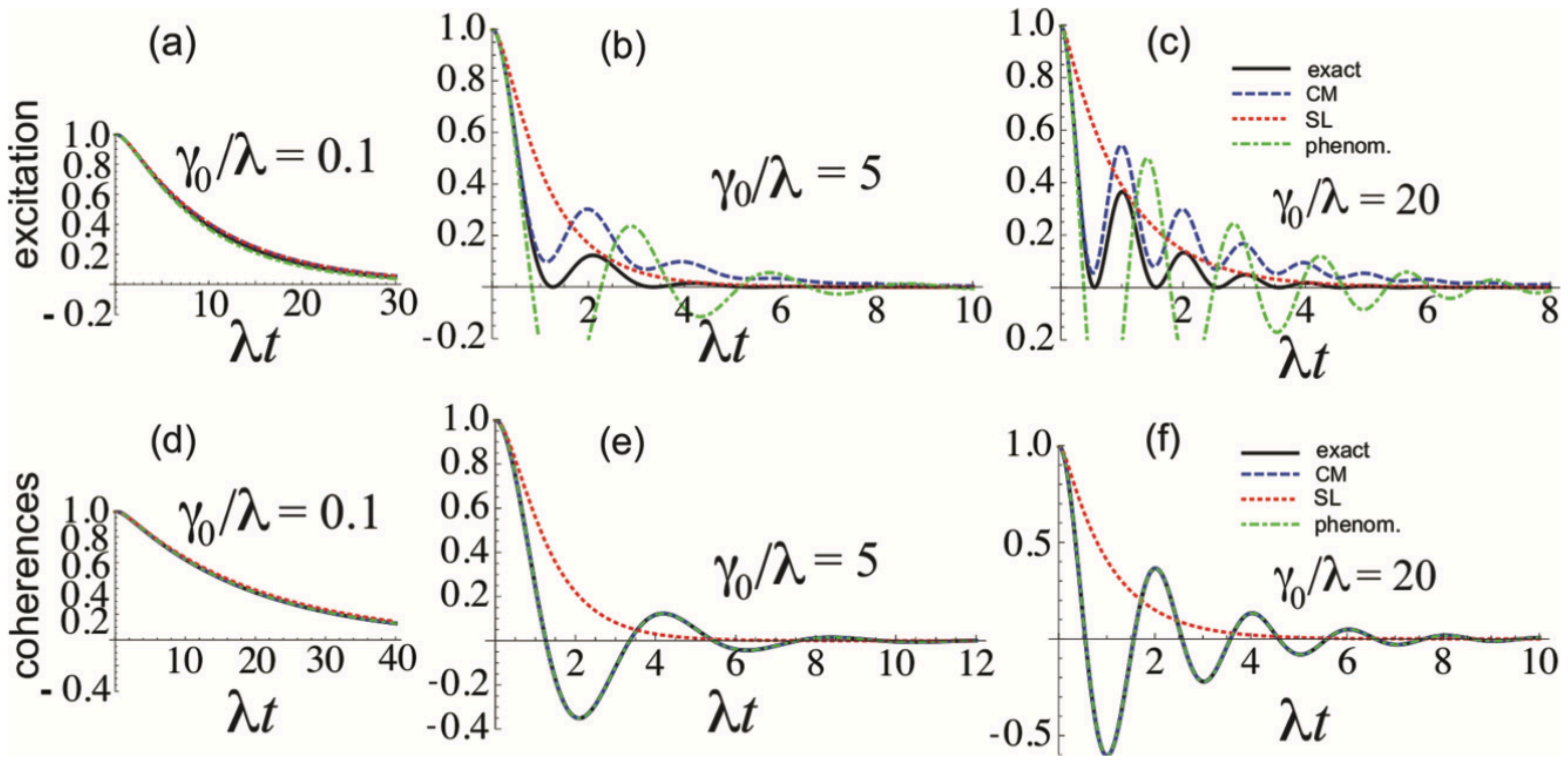
Partial SWAP map (CPTP)

$$\mathcal{U}_j\rho = e^{-iH_{I_j}\tau}\rho e^{iH_{I_j}\tau} \qquad p = e^{-\gamma\tau}$$

Continuous `time' limit: $t = n\tau \qquad \tau \to 0 \quad n \to \infty$

$$\dot{\rho}_S(t) = \gamma \int_0^t ds\, e^{-\gamma s}\Phi_s \dot{\rho}_S(t-s) + e^{-\gamma t}\dot{\Phi}_t[\rho_S(0)]$$

Two-level atom + vacuum reservoir

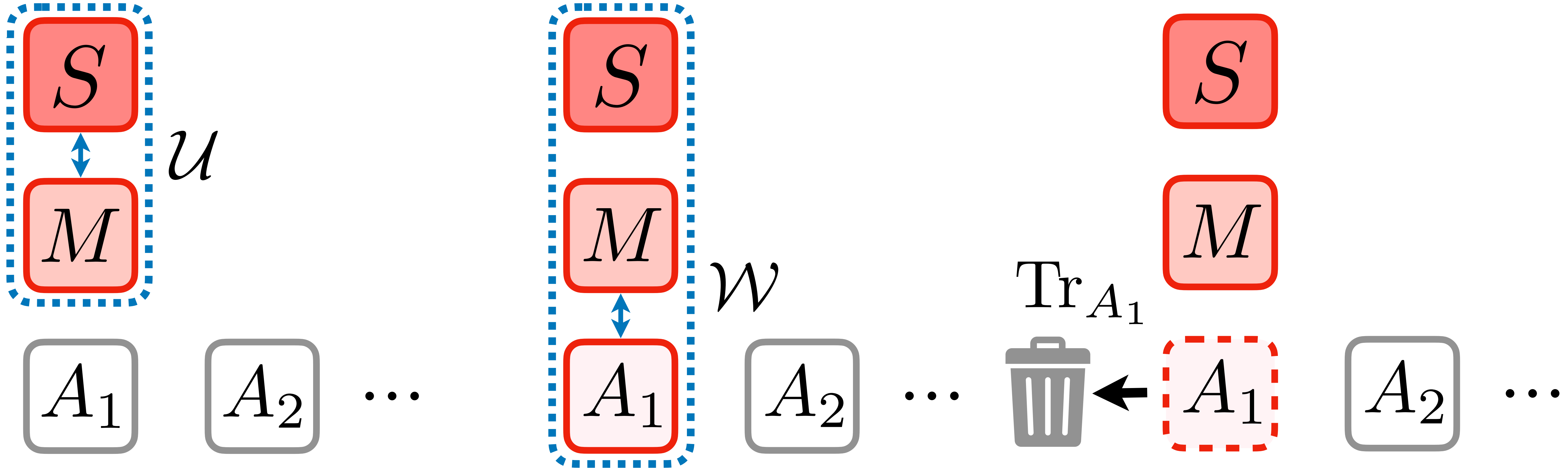


F. Ciccarello et al, PRA 87, 040103(R) (2013)

Composite CM

- Open system S includes additional memory ancilla M
- $S-M$ interaction responsible for memory effects in reduced system dynamics.

$$\rho(n) = \mathcal{W}(\mathcal{U}\cdots\mathcal{W}(\mathcal{U}(\rho_{SM}(0)\otimes\sigma\otimes\ldots\otimes\sigma))\cdots) \qquad \rho_{SM}(n) = \mathrm{Tr}_{A_1\ldots A_n}[\rho(n)]$$



S. Lorenzo et al., PRA 96, 32107 (2017)

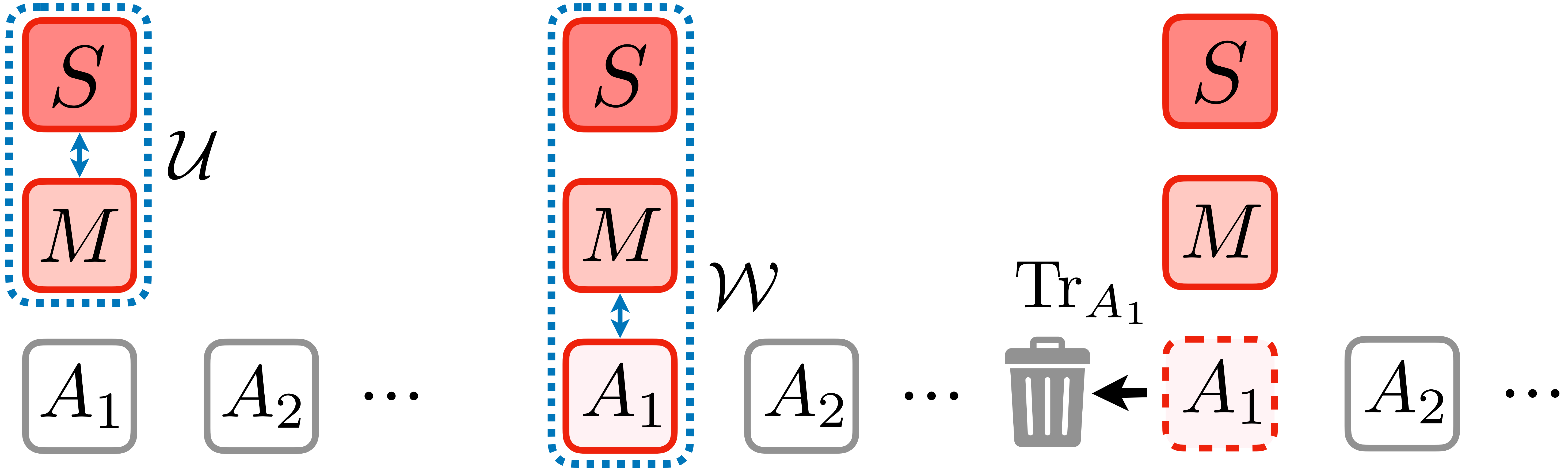
Composite CM

- Open system S includes additional memory ancilla M
- $S-M$ interaction responsible for memory effects in reduced system dynamics.

$$\rho_{SM}(n) = \Phi' \circ \ldots \circ \Phi'[\rho_{SM}(0)]$$

$$\Phi'[\rho] = \mathrm{Tr}_A[\mathcal{W}(\mathcal{U}(\rho \otimes \sigma))]$$

CPTP map for enlarged system



S. Lorenzo et al., PRA 96, 32107 (2017)

Composite CM

Stellenbosch University Iyunivesithi Universiteit

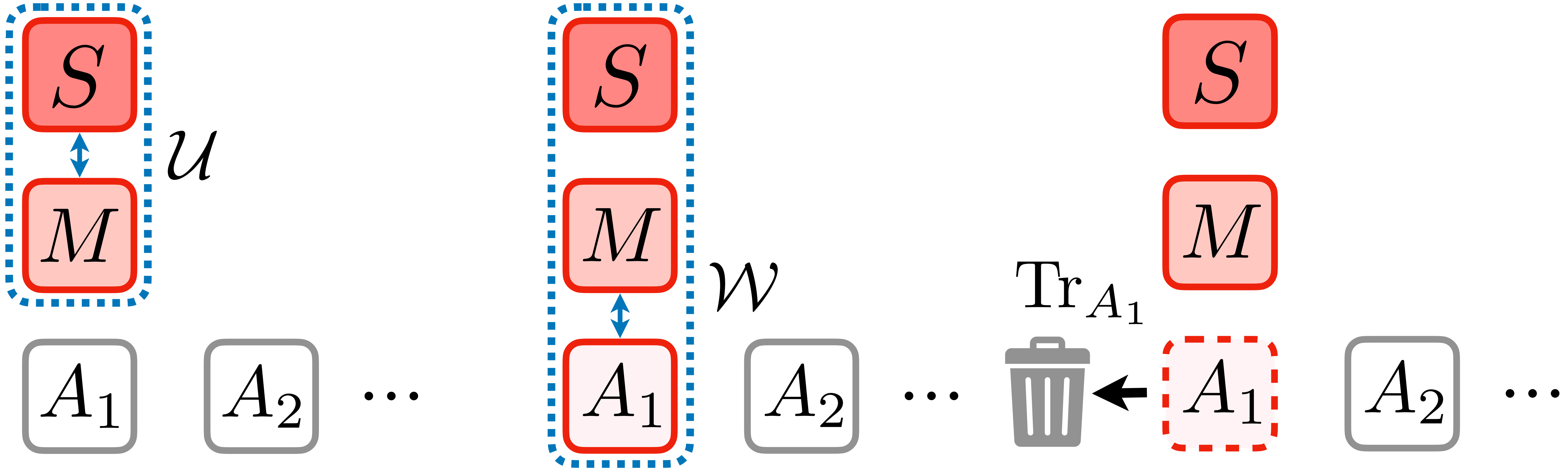
- Open system S includes additional memory ancilla M
- $S-M$ interaction responsible for memory effects in reduced system dynamics.

$$\rho_{SM}(n) = \Phi' \circ \ldots \circ \Phi'[\rho_{SM}(0)] \qquad \Phi'[\rho] = \mathrm{Tr}_A[\mathcal{W}(\mathcal{U}(\rho \otimes \sigma))]$$

$$\rho_S(n) = \mathrm{Tr}_M \mathrm{Tr}_A[\mathcal{U}'(\rho_{SM}(n-1) \otimes \sigma)] := \Phi_n[\rho_S(0)]$$

system map generally indivisible

$$\Phi_n \neq \Phi_{n-m}\Phi_m$$



S. Lorenzo et al., PRA 96, 32107 (2017)

Composite CM

Stellenbosch UNIVERSITY IYUNIVESITHI UNIVERSITEIT

Two-level atom + vacuum reservoir:

$$U = e^{-i\tau(V_{SM}+H_M)} \qquad W_j = e^{-i\tau H_{I_j}} \qquad S, M, R \text{ qubits}$$

where $V_{SM} = g(\sigma_+\sigma_-^M + \sigma_-\sigma_+^M)$ $\quad H_{I_j} = \kappa(\sigma_+^M\sigma_-^{A_j} + \sigma_-^M\sigma_+^{A_j})$ $\quad H_M = \Delta\sigma_+^M\sigma_-^M$

S. Lorenzo et al., PRA 96, 32107 (2017)

Composite CM

Stellenbosch UNIVERSITY IYUNIVESITHI UNIVERSITEIT

Two-level atom + vacuum reservoir:

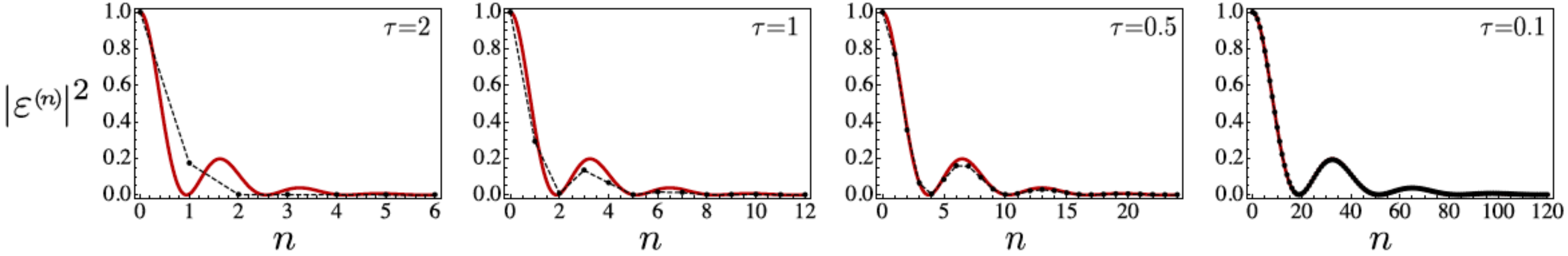
$$U = e^{-i\tau(V_{SM}+H_M)} \qquad W_j = e^{-i\tau H_{I_j}} \qquad S, M, R \text{ qubits}$$

where $V_{SM} = g(\sigma_+\sigma_-^M + \sigma_-\sigma_+^M)$ $\quad H_{I_j} = \kappa(\sigma_+^M\sigma_-^{A_j} + \sigma_-^M\sigma_+^{A_j})$ $\quad H_M = \Delta\sigma_+^M\sigma_-^M$

$$|\psi_n\rangle = \varepsilon^{(n)}|e,0,\vec{0}\rangle + \beta^{(n)}|g,1,\vec{0}\rangle + \sum_{j=1}^{n}\lambda_j^{(n)}\sigma_+^{A_j}|g,0,\vec{0}\rangle$$

initially excited atom $|\varepsilon^{(0)}|^2 = 1$

*see Marco's lectures



$\Delta = 0$

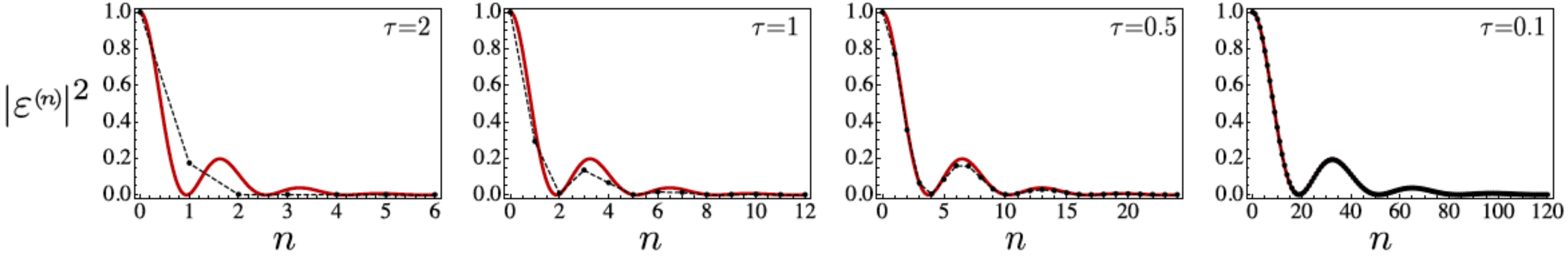
S. Lorenzo et al., PRA 96, 32107 (2017)

Composite CM

Two-level atom + vacuum reservoir:

Continuous time limit: $t = n\tau$ $\tau \to 0$ $n \to \infty$ $g\tau \ll \kappa\tau \ll 1$ $\gamma = \kappa^2\tau$

$$\varepsilon^{(n)} \to \varepsilon(t) \qquad \beta^{(n)} \to \beta(t) \qquad \lambda_j^{(n)} \to \lambda_j(t)$$



$\Delta = 0$

S. Lorenzo et al., PRA 96, 32107 (2017)

Composite CM

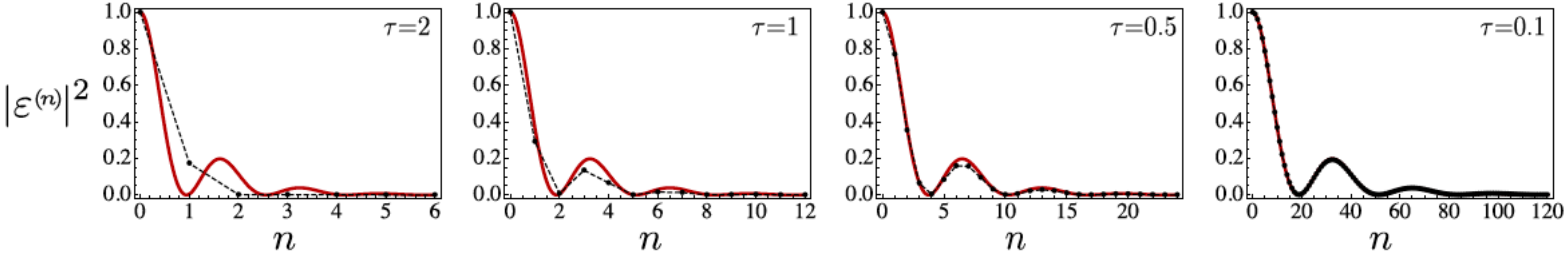
Two-level atom + vacuum reservoir:

Continuous time limit: $t = n\tau \qquad \tau \to 0 \quad n \to \infty \qquad g\tau \ll \kappa\tau \ll 1 \qquad \gamma = \kappa^2\tau$

$$\varepsilon^{(n)} \to \varepsilon(t) \qquad \beta^{(n)} \to \beta(t) \qquad \lambda_j^{(n)} \to \lambda_j(t)$$

$$|\varepsilon(t)|^2 = e^{-\gamma t/2}\left[\cosh(\Omega t) + \frac{\gamma}{4\Omega}\sinh(\Omega t)\right]^2 \qquad \Omega = \frac{1}{2}\sqrt{(\gamma/2)^2 - 4g^2}$$

*equivalent to $|G(t)|^2$ in continuous time limit



$\Delta = 0$

S. Lorenzo et al., PRA 96, 32107 (2017)

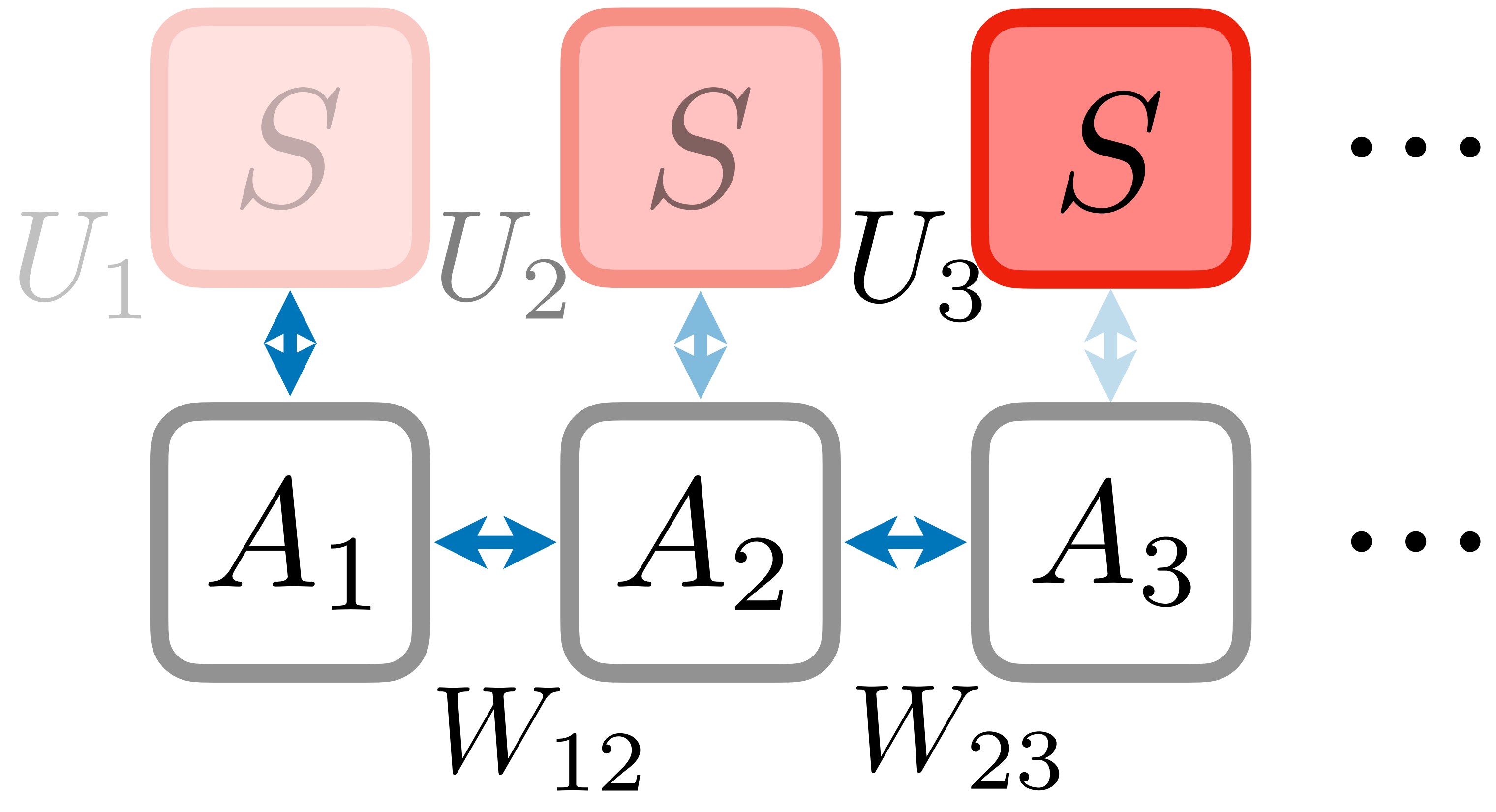
Mapping

Ancilla-ancilla collisions $\Longleftrightarrow$ composite collision model

$$\rho_S(n) = \mathrm{Tr}_{A_1..A_{n+1}}[\mathcal{W}_{n+1,n}\mathcal{U}_n(...\mathcal{W}_{2,1}\mathcal{U}_1(\rho_S(0)\otimes\sigma)\otimes\sigma)...\otimes\sigma)]$$

$$= \mathrm{Tr}_{A_{n+1}}\Phi'^n[\rho_S(0)\otimes\sigma]$$

$$\Phi'[\rho] = \mathrm{Tr}_{A_1}[\mathcal{W}_{2,1}\mathcal{U}_1(\rho\otimes\sigma)]$$



F. Ciccarello et al., PR 954, 1 (2022)   S. Campbell et al., PRA 98, 12142 (2018)

Mapping

Ancilla-ancilla collisions $\Longleftrightarrow$ composite collision model

$$\rho_S(n) = \mathrm{Tr}_{A_1..A_{n+1}}[\mathcal{W}_{n+1,n}\mathcal{U}_n(...\mathcal{W}_{2,1}\mathcal{U}_1(\rho_S(0)\otimes\sigma)\otimes\sigma)...\otimes\sigma)]$$

$$= \mathrm{Tr}_{A_{n+1}}\Phi'^n[\rho_S(0)\otimes\sigma]$$

$$\Phi'[\rho] = \mathrm{Tr}_{A_1}[\mathcal{W}_{2,1}\mathcal{U}_1(\rho\otimes\sigma)]$$

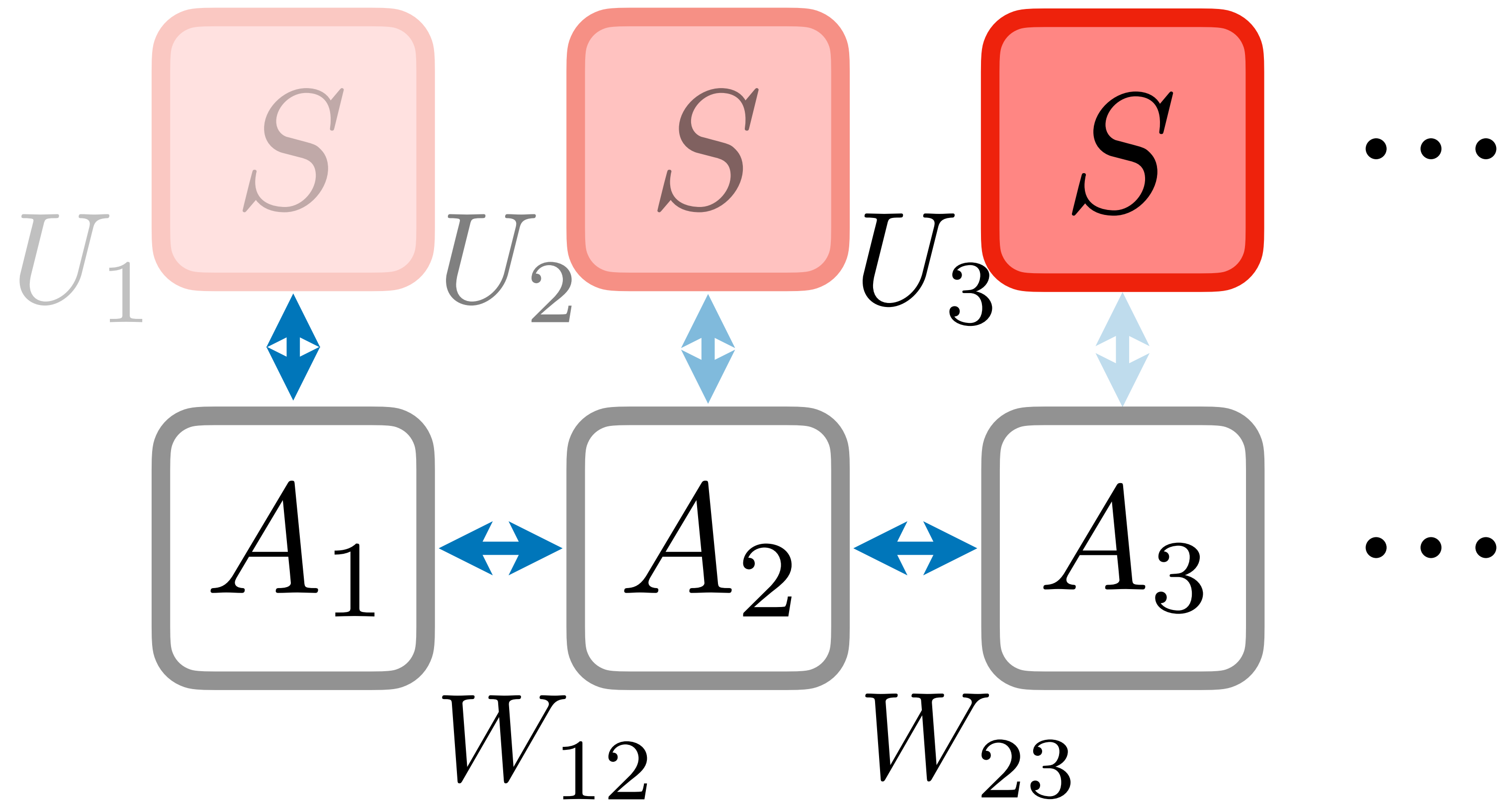
SWAP map:

$$\mathcal{S}_{k,l}\mathcal{W}_{k,l} = \mathcal{W}_{k,l}\mathcal{S}_{k,l} \qquad \mathcal{S}_{k,l}^2 = \mathcal{I}$$

$$\Rightarrow \quad \Phi'[\rho] = \mathrm{Tr}_{A_1}[(\mathcal{S}_{2,1}^2)\mathcal{W}_{2,1}\mathcal{U}_1(\rho\otimes\sigma)]$$

$$= \mathrm{Tr}_{A_1}[(\mathcal{S}_{2,1})\mathcal{W}'_{2,1}\mathcal{U}_1(\rho\otimes\sigma)]$$

$$= \mathrm{Tr}_{A_2}[\mathcal{W}'_{2,1}\mathcal{U}_1(\rho\otimes\sigma)]$$



F. Ciccarello et al., PR 954, 1 (2022)   S. Campbell et al., PRA 98, 12142 (2018)

Mapping

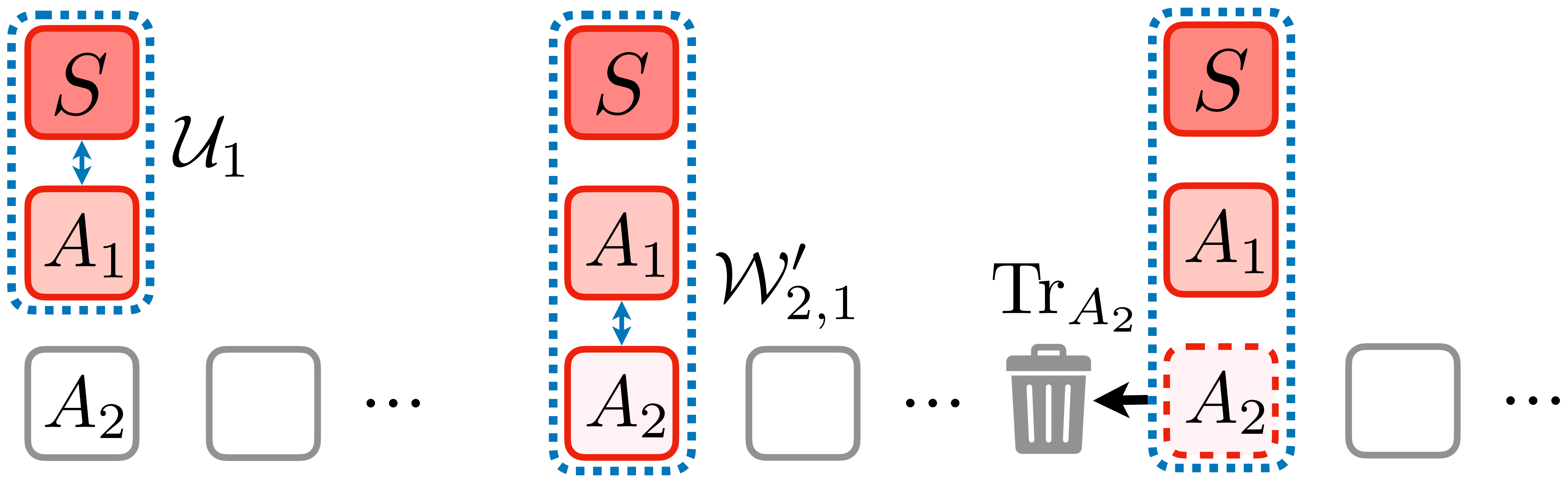
Ancilla-ancilla collisions $\Longleftrightarrow$ composite collision model

$$\rho_S(n) = \mathrm{Tr}_{A_1..A_{n+1}}[\mathcal{W}_{n+1,n}\mathcal{U}_n(...\mathcal{W}_{2,1}\mathcal{U}_1(\rho_S(0)\otimes\sigma)\otimes\sigma)...\otimes\sigma)]$$

$$= \mathrm{Tr}_{A_{n+1}}\Phi'^n[\rho_S(0)\otimes\sigma]$$

$$\Phi'[\rho] = \mathrm{Tr}_{A_2}[\mathcal{W}'_{2,1}\mathcal{U}_1(\rho\otimes\sigma)]$$

$$\mathcal{W}'_{2,1} = \mathcal{S}_{2,1}\mathcal{W}_{2,1}$$



F. Ciccarello et al., PR 954, 1 (2022) S. Campbell et al., PRA 98, 12142 (2018)

Summary

Stellenbosch UNIVERSITY IYUNIVESITHI UNIVERSITEIT

- Markov approximation - separation of time scales.
- Definition of Markovianity for classical stochastic processes not immediately generalisable to quantum domain (inconsistency with Kolmogorov axioms).
- Intrinsic definition of quantum Markovianity - divisibility and distinguishability.
- Collision models - platform to simulate non-Markovian open quantum dynamics (e.g., spontaneous emission of two-level atom).