33rd Chris Engelbrecht Summer School

April 2025

Introduction to Quantum Thermodynamics

Lecture 2

Camille L Latune
camille.lombard-latune@u-bourgogne.fr

Lecture II:
Quantum heat engines, entropy production, and experimental implementations

I. Quantum Otto Engine
II. Irreversibility, entropy production and second law for quantum systems
III. Quantum Carnot Engine
IV. The question of quantum advantages
V. Experimental implementation of quantum autonomous refrigerator
VI. Opening: work cost of quantum measurements

I. Quantum Otto Engine

I. Quantum Otto Engine

Quantum Otto cycle with a qubit (spin-1/2)

Observation: for simplicity, we define the ground state energy of the spin to be equal to 0

→ $H = \hbar\omega|e\rangle\langle e|$ (in the following, $\hbar = 1$) with, for a spin-$1/2$, $\omega = \gamma|\vec{B}|$

→ the energy splitting is controled by the intensity of the applied magnetic field

→ **Thermal state at temperature T, or equivalently at inverse temperature** $\beta := 1/(k_B T)$ **:**

$$\rho^{\mathrm{th}}(\beta,\omega) := \frac{1}{Z(\beta,\omega)} e^{-\beta H} \quad \text{with} \quad Z(\beta,\omega) = \mathrm{Tr}\left[e^{-\beta H}\right] = 1 + e^{-\hbar\omega\beta}$$

→ **In the basis** $\{|e\rangle, |g\rangle\}$ **:**

$$\rho^{\mathrm{th}}(\beta,\omega) = \frac{1}{1+e^{-\hbar\omega\beta}}\left(e^{-\hbar\omega\beta}|e\rangle\langle e| + |g\rangle\langle g|\right)$$

I. Quantum Otto Engine

Quantum Otto cycle with a qubit (spin-1/2)

Observation: for simplicity, we define the ground state energy of the spin to be equal to 0

→ $H = \hbar\omega|e\rangle\langle e|$ with, for a spin-$1/2$, $\omega = \gamma|\vec{B}|$

(in the following, $\hbar = 1$)

→ the energy splitting is controled by the intensity of the applied magnetic field

→ **Thermal state at temperature T, or equivalently at inverse temperature** $\beta := 1/(k_B T)$ **:**

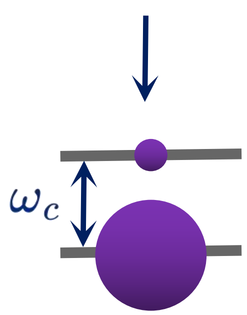
$$\rho^{\mathrm{th}}(\beta,\omega) := \frac{1}{Z(\beta,\omega)} e^{-\beta H}$$

with $Z(\beta,\omega) = \mathrm{Tr}\left[e^{-\beta H}\right] = 1 + e^{-\hbar\omega\beta}$

→ **In the basis** $\{|e\rangle, |g\rangle\}$ **:**

$$\rho^{\mathrm{th}}(\beta,\omega) = \frac{1}{1+e^{-\hbar\omega\beta}}\left(e^{-\hbar\omega\beta}|e\rangle\langle e| + |g\rangle\langle g|\right)$$

Initial state of the cycle



$\rho^{\mathrm{th}}(\beta_c,\omega_c)$

Thermal state at T_c **(inverse temperature** $\beta_c = 1/(k_B T_c)$ **)**

I. Quantum Otto Engine

Quantum Otto cycle with a qubit (spin-1/2)

Observation: for simplicity, we define the ground state energy of the spin to be equal to 0

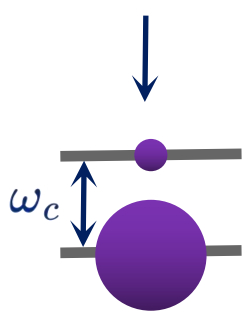
→ $H = \hbar\omega |e\rangle\langle e|$ with, for a spin-1/2, $\omega = \gamma|\vec{B}|$ (in the following, $\hbar = 1$)

→ the energy splitting is controled by the intensity of the applied magnetic field

→ **Thermal state at temperature T, or equivalently at inverse temperature $\beta := 1/(k_B T)$:** $\rho^{\mathrm{th}}(\beta,\omega) := \dfrac{1}{Z(\beta,\omega)} e^{-\beta H}$ with $Z(\beta,\omega) = \mathrm{Tr}\left[e^{-\beta H}\right] = 1 + e^{-\hbar\omega\beta}$

→ **In the basis $\{|e\rangle, |g\rangle\}$:** $\rho^{\mathrm{th}}(\beta,\omega) = \dfrac{1}{1+e^{-\hbar\omega\beta}}\left(e^{-\hbar\omega\beta}|e\rangle\langle e| + |g\rangle\langle g|\right)$

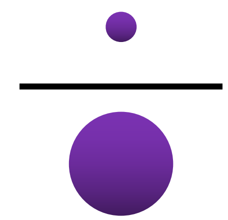
Initial state of the cycle



$\rho^{\mathrm{th}}(\beta_c,\omega_c)$

Thermal state at T_c (inverse temperature $\beta_c = 1/(k_B T_c)$)

→ **Observation: the size of each sphere represents the population associated to each energy level** →

“  $= e^{-\hbar\omega_c/k_B T_c} = e^{-\hbar\omega_c\beta_c}$ ”

I. Quantum Otto Engine

Quantum Otto cycle with a qubit (spin-1/2)

Observation: for simplicity, we define the ground state energy of the spin to be equal to 0

→ $H = \hbar\omega|e\rangle\langle e|$ (in the following, $\hbar = 1$) with, for a spin-1/2, $\omega = \gamma|\vec{B}|$

→ the energy splitting is controled by the intensity of the applied magnetic field

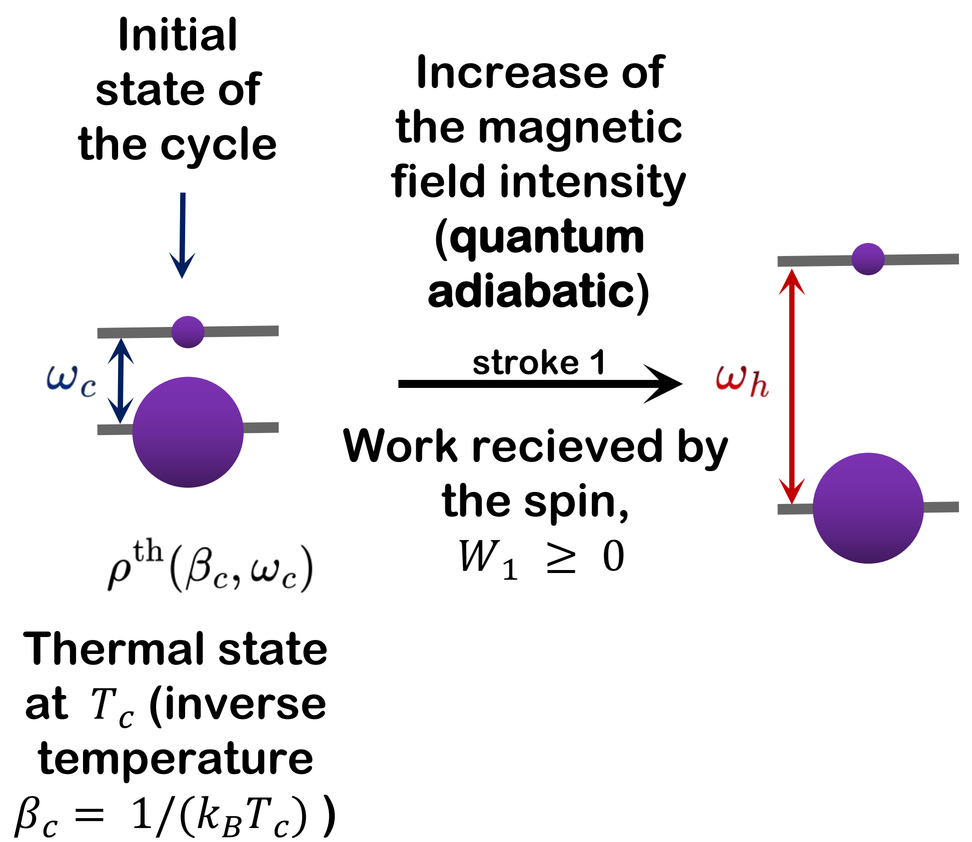
→ **Thermal state at temperature T, or equivalently at inverse temperature** $\beta := 1/(k_B T)$ **:**

$$\rho^{\text{th}}(\beta,\omega) := \frac{1}{Z(\beta,\omega)} e^{-\beta H}$$

with $Z(\beta,\omega) = \text{Tr}\left[e^{-\beta H}\right] = 1 + e^{-\hbar\omega\beta}$

→ **In the basis** $\{|e\rangle, |g\rangle\}$ **:**

$$\rho^{\text{th}}(\beta,\omega) = \frac{1}{1+e^{-\hbar\omega\beta}}\left(e^{-\hbar\omega\beta}|e\rangle\langle e| + |g\rangle\langle g|\right)$$



Quantum Otto cycle with a qubit (spin-1/2)

Observation: for simplicity, we define the ground state energy of the spin to be equal to 0

→ $$H = \hbar\omega |e\rangle\langle e| \quad \text{with, for a spin-1/2,} \quad \omega = \gamma|\vec{B}|$$

(in the following, $\hbar = 1$)

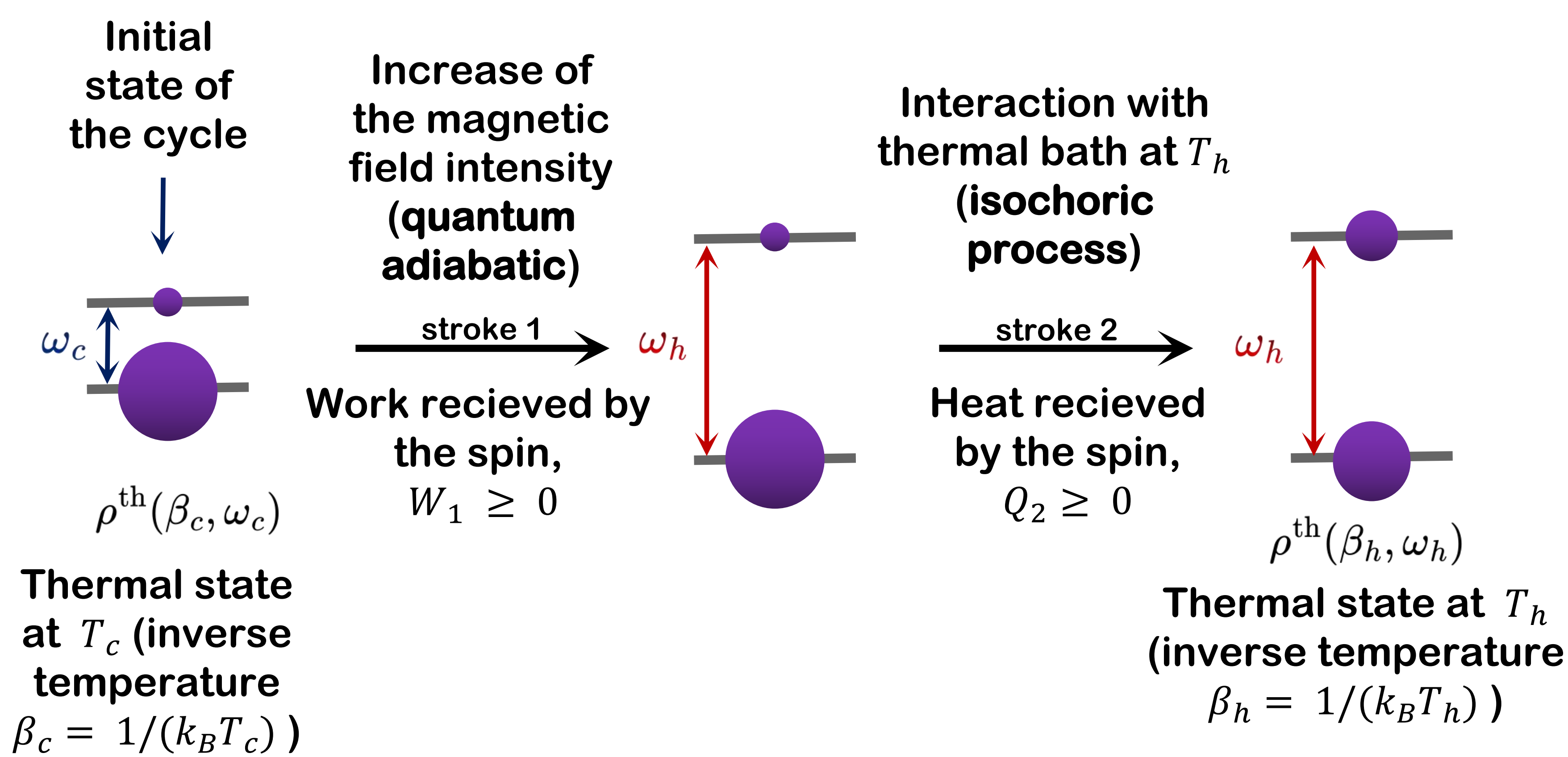
→ the energy splitting is controled by the intensity of the applied magnetic field

→ **Thermal state at temperature T, or equivalently at inverse temperature $\beta := 1/(k_B T)$:**

$$\rho^{\text{th}}(\beta,\omega) := \frac{1}{Z(\beta,\omega)} e^{-\beta H} \quad \text{with} \quad Z(\beta,\omega) = \text{Tr}\left[e^{-\beta H}\right] = 1 + e^{-\hbar\omega\beta}$$

→ **In the basis $\{|e\rangle, |g\rangle\}$:**

$$\rho^{\text{th}}(\beta,\omega) = \frac{1}{1+e^{-\hbar\omega\beta}}\left(e^{-\hbar\omega\beta}|e\rangle\langle e| + |g\rangle\langle g|\right)$$



I. Quantum Otto Engine

Quantum Otto cycle with a qubit (spin-1/2)

Observation: for simplicity, we define the ground state energy of the spin to be equal to 0

→ $H = \hbar\omega |e\rangle\langle e|$ with, for a spin-1/2, $\omega = \gamma|\vec{B}|$

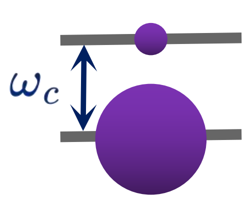
(in the following, $\hbar = 1$)

→ the energy splitting is controled by the intensity of the applied magnetic field

→ **Thermal state at temperature T, or equivalently at inverse temperature $\beta := 1/(k_B T)$:** $\rho^{\mathrm{th}}(\beta,\omega) := \frac{1}{Z(\beta,\omega)} e^{-\beta H}$ with $Z(\beta,\omega) = \mathrm{Tr}\left[e^{-\beta H}\right] = 1 + e^{-\hbar\omega\beta}$

→ **In the basis $\{|e\rangle, |g\rangle\}$:** $\rho^{\mathrm{th}}(\beta,\omega) = \frac{1}{1+e^{-\hbar\omega\beta}}\left(e^{-\hbar\omega\beta}|e\rangle\langle e| + |g\rangle\langle g|\right)$

Initial state of the cycle



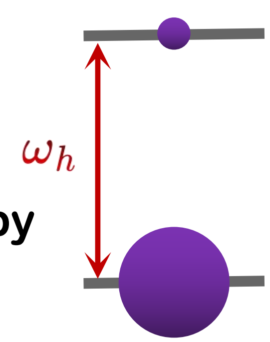
$\rho^{\mathrm{th}}(\beta_c,\omega_c)$

Thermal state at T_c (inverse temperature $\beta_c = 1/(k_B T_c)$)

Increase of the magnetic field intensity (quantum adiabatic)

stroke 1 →

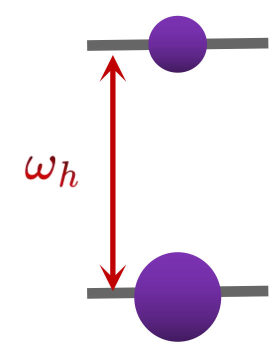
Work recieved by the spin, $W_1 \geq 0$



Interaction with thermal bath at T_h (isochoric process)

stroke 2 →

Heat recieved by the spin, $Q_2 \geq 0$



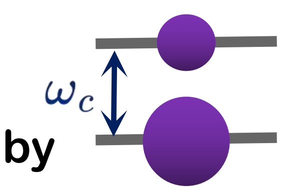
$\rho^{\mathrm{th}}(\beta_h,\omega_h)$

Thermal state at T_h (inverse temperature $\beta_h = 1/(k_B T_h)$)

Decrease of the magnetic field (quantum adiabatic)

stroke 3 →

Work provided by the spin, $W_3 \leq 0$



I. Quantum Otto Engine

Quantum Otto cycle with a qubit (spin-1/2)

Observation: for simplicity, we define the ground state energy of the spin to be equal to 0

→ $H = \hbar\omega|e\rangle\langle e|$ with, for a spin-1/2, $\omega = \gamma|\vec{B}|$

(in the following, $\hbar = 1$)

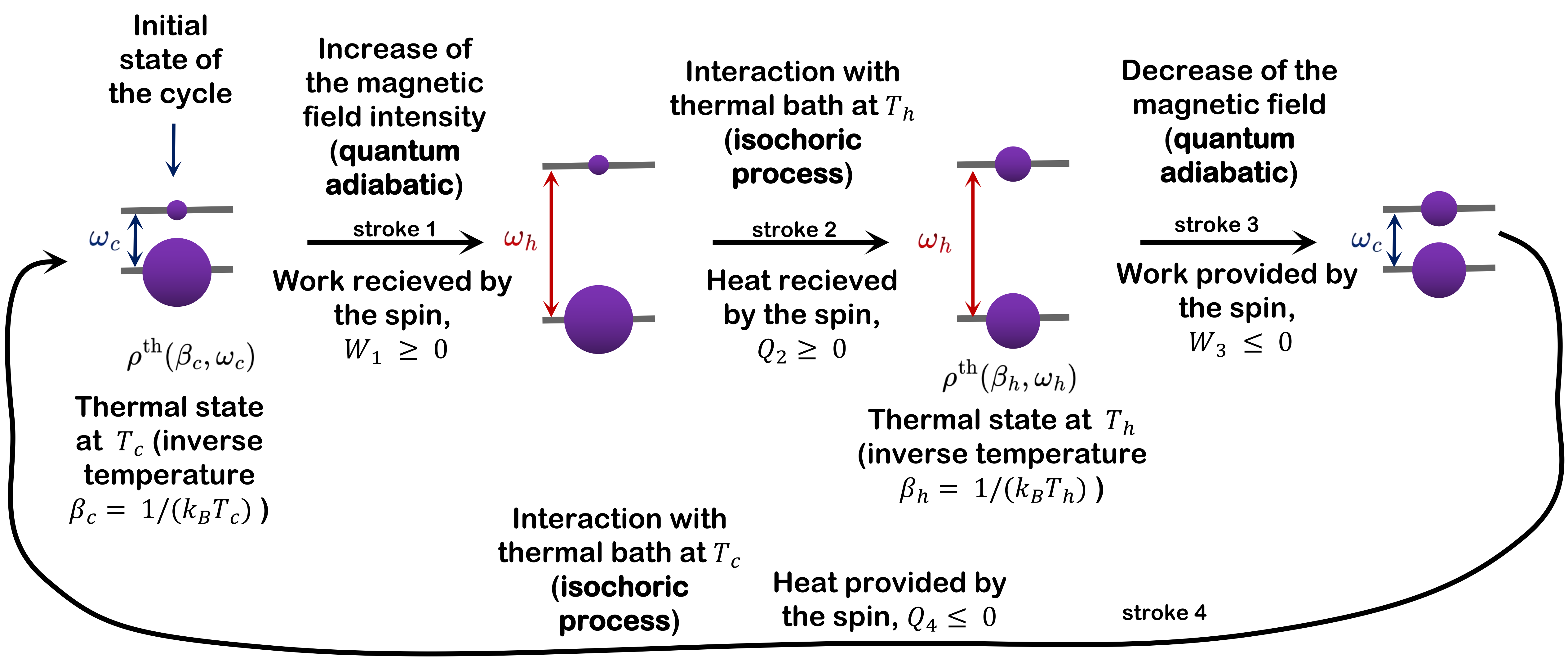
→ the energy splitting is controled by the intensity of the applied magnetic field

→ **Thermal state at temperature T, or equivalently at inverse temperature $\beta := 1/(k_B T)$:**

$$\rho^{\mathrm{th}}(\beta,\omega) := \frac{1}{Z(\beta,\omega)} e^{-\beta H} \quad \text{with} \quad Z(\beta,\omega) = \mathrm{Tr}\left[e^{-\beta H}\right] = 1 + e^{-\hbar\omega\beta}$$

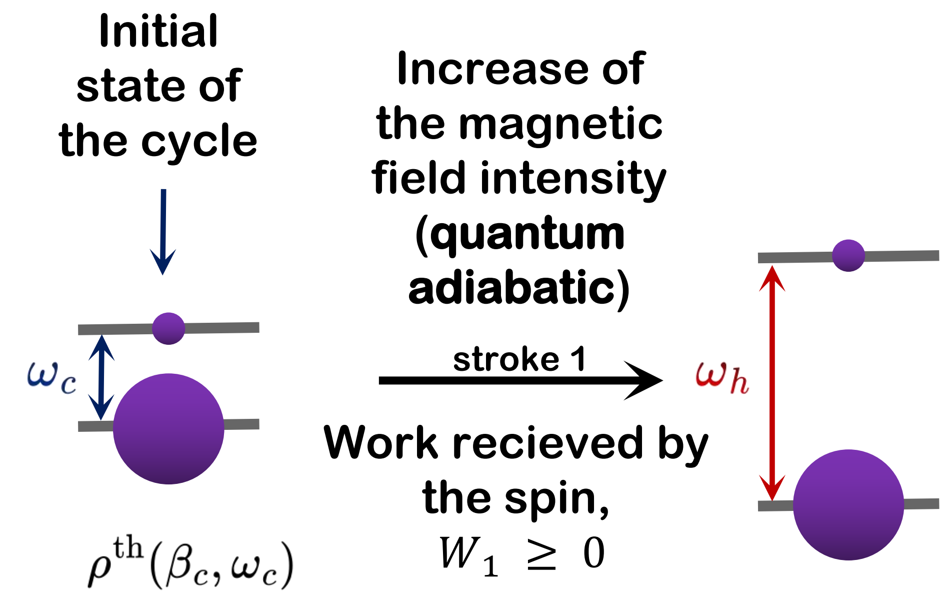
→ **In the basis $\{|e\rangle, |g\rangle\}$:**

$$\rho^{\mathrm{th}}(\beta,\omega) = \frac{1}{1+e^{-\hbar\omega\beta}}\left(e^{-\hbar\omega\beta}|e\rangle\langle e| + |g\rangle\langle g|\right)$$



I. Quantum Otto Engine

Stroke 1: from $t = 0$ to $t = t_1$



- **Hamiltonian during stroke 1:** $$H(t) = \hbar\omega_1(t)|e\rangle\langle e|$$

with $\omega_1(t)$ being a function such that $\omega_1(0) = \omega_c$ and $\omega_1(t_1) = \omega_h$

- **Evolution operator during stroke 1:**

$$U_1(t) = \underbrace{\mathcal{T} e^{\frac{i}{\hbar}\int_0^t du H(u)}}_{\text{Dyson series (see reminder in Section II)}} = e^{-i\left(\int_0^t du\omega_1(u)\right)|e\rangle\langle e|}$$

Because $[H(t), H(t')] = [\omega_1(t)|e\rangle\langle e|, \omega_1(t')|e\rangle\langle e|] = \omega_1(t)\omega_1(t')[|e\rangle\langle e|, |e\rangle\langle e|] = 0, \text{ for all } t, t'$

$$= e^{-i\left(\int_0^t du\omega_1(u)\right)}|e\rangle\langle e| + |g\rangle\langle g|$$

$$= e^{-i\Omega_1(t)}|e\rangle\langle e| + |g\rangle\langle g| \qquad \text{with } \Omega_1(t) = \int_0^t du\omega_1(u)$$

I. Quantum Otto Engine

- **Evolution of the qubit state during stroke 1:**

$$\rho(t) = U_1(t)\rho^{\rm th}(\beta_c,\omega_c)U_1(t)^\dagger$$

$$= \left[e^{-i\Omega_1(t)}|e\rangle\langle e| + |g\rangle\langle g|\right]\rho^{\rm th}(\beta_c,\omega_c)\left[e^{i\Omega_1(t)}|e\rangle\langle e| + |g\rangle\langle g|\right]$$

$$= \left[e^{-i\Omega_1(t)}|e\rangle\langle e| + |g\rangle\langle g|\right]\frac{1}{1+e^{-\hbar\omega_c\beta_c}}\left(e^{-\hbar\omega_c\beta_c}|e\rangle\langle e| + |g\rangle\langle g|\right)\left[e^{i\Omega_1(t)}|e\rangle\langle e| + |g\rangle\langle g|\right]$$

$$= \frac{1}{1+e^{-\hbar\omega_c\beta_c}}\left(e^{-\hbar\omega_c\beta_c}|e\rangle\langle e| + |g\rangle\langle g|\right)$$

$$= \rho^{\rm th}(\beta_c,\omega_c)$$

$\rightarrow$ **The state of the qubit remains constant under such evolution!**

$\rightarrow$ **We could have predicted that because**

$$[\rho^{\rm th}(\beta_c,\omega_c), H(t)] = 0 \ \text{ for all } t \in [0;t_1]$$

- **Work exchange during stroke 1:**

$$W_1 := \int_0^{t_1} dt \mathrm{Tr}[\rho(t)\dot{H}(t)] = \hbar\int_0^{t_1} dt\mathrm{Tr}[\rho^{\rm th}(\beta_c,\omega_c)\dot{\omega}_1(t)|e\rangle\langle e|]$$

$$= \hbar\mathrm{Tr}[\rho^{\rm th}(\beta_c,\omega_c)|e\rangle\langle e|]\int_0^{t_1} dt\dot{\omega}_1(t)$$

$$= \hbar\mathrm{Tr}[\rho^{\rm th}(\beta_c,\omega_c)|e\rangle\langle e|]\Big(\omega_1(t_1) - \omega_1(0)\Big)$$

with $\quad p_e(\beta_c,\omega_c) := \dfrac{e^{-\hbar\omega_c\beta_c}}{1+e^{-\hbar\omega_c\beta_c}}$

$$= \hbar(\omega_h - \omega_c)p_e(\beta_c,\omega_c) \geq 0$$

since $\omega_h \geq \omega_c$

I. Quantum Otto Engine

- **Heat exchange during stroke 1:**

$$Q_1 := \int_0^{t_1} dt \mathrm{Tr}[\dot{\rho}(t) H(t)] = \int_0^{t_1} dt \mathrm{Tr}\Big[\Big(- \frac{i}{\hbar}[H(t), \rho(t)]\Big) H(t)\Big]$$

$$= - \frac{i}{\hbar} \int_0^{t_1} dt \mathrm{Tr}\Big[\rho(t)[H(t), H(t)]\Big]$$

$$= 0$$

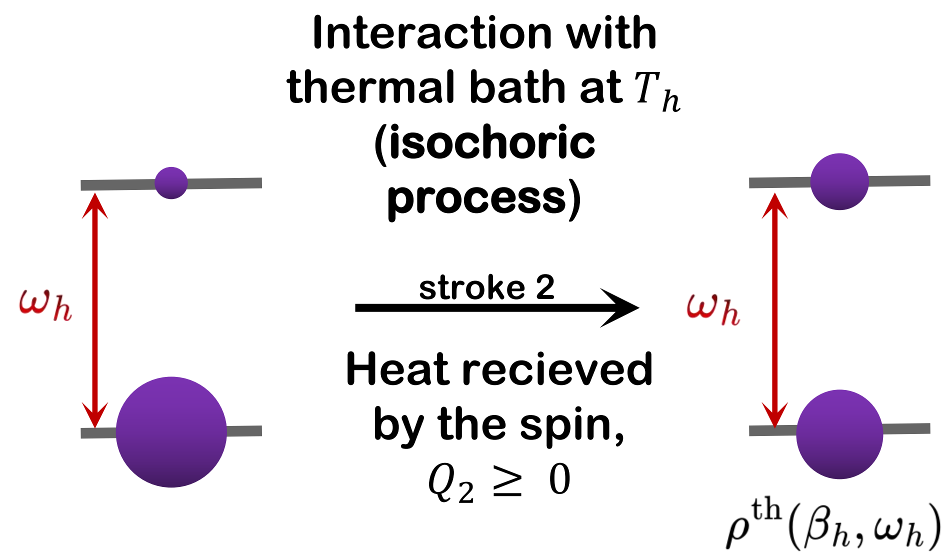
Thanks to the invariance of the Trace under cyclic permutations

$\rightarrow$ **This was expected because during the stroke 1, the qubit evolves unitarily, with no interaction with any bath, so no heat exchanges were expected.**

$\longrightarrow$ **Similar computation for stroke 3**

I. Quantum Otto Engine

Stroke 2: from $t = t_1$ to $t = t_2$



- **Hamiltonian during stroke 2:** $H = \hbar\omega_h |e\rangle\langle e|$ **(constant)**

- **Assuming weak coupling with the hot bath, so that the Born and Markov (memoryless dynamics) (see Chapter 4 of Quantum Physics Lectures) are valid and the dynamics during stroke 2 is given by the following GKLS master equation:**

$$\begin{aligned}\dot{\rho}(t) = -\frac{i}{\hbar}[H, \rho(t)] &+ \gamma_+\Big(\sigma_+\rho(t)\sigma_- - \frac{1}{2}\{\sigma_-\sigma_+, \rho(t)\}\Big) \\ &+ \gamma_-\Big(\sigma_-\rho(t)\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho(t)\}\Big) \\ &+ \frac{\gamma_z}{2}\Big(\sigma_z\rho(t)\sigma_z - \rho(t)\Big)\end{aligned}$$

with $\sigma_+ := |e\rangle\langle g|$, $\sigma_- := |g\rangle\langle e|$ and $\dfrac{\gamma_+}{\gamma_-} = e^{-\hbar\omega_h\beta_h}$

(Additional information (not useful here): the coefficients γ_+, γ_- and γ_z are actually defined from $\gamma(\omega) := \int_{-\infty}^{\infty} du e^{-i\omega u}\mathrm{Tr}[\rho_{B_h}(0) X^I_{B_h}(u) X_{B_h}]$ when evaluated at $\omega = \omega_h$, $\omega = -\omega_h$ and $\omega = 0$, respectively)

I. Quantum Otto Engine

- **Evolution of the qubit state during stroke 2:**

As already shown in Chapter 4 of Quantum Physics Lecture, and recalled in the beginning of these notes, the dynamics of the excited population is given by

$$\frac{d}{dt}p_e(t) = \frac{d}{dt}\langle e|\rho_S(t)|e\rangle = \langle e|\frac{d}{dt}\rho_S(t)|e\rangle =\ldots= \gamma_+ p_g(t) - \gamma_- p_e(t) = -(\gamma_+ + \gamma_-)p_e(t) + \gamma_+$$

(using the master equation of the previous page) $\qquad p_e(t) + p_g(t) = 1$

$$\Longrightarrow p_e(t) = e^{-t/T_1}\left(p_e(0) - \frac{\gamma_+}{\gamma_+ + \gamma_-}\right) + \frac{\gamma_+}{\gamma_+ + \gamma_-} \quad \textbf{with} \quad \frac{1}{T_1} := \gamma_+ + \gamma_-$$

→ **Remembering that the stroke takes place between $[t_1; t_2]$, we replace 0 by t_1 and t by $t - t_1$ in the right-hand side, implying, for all t in $[t_1; t_2]$,**

$$p_e(t) = e^{-(t-t_1)/T_1}\left(p_e(t_1) - \frac{\gamma_+}{\gamma_+ + \gamma_-}\right) + \frac{\gamma_+}{\gamma_+ + \gamma_-}$$

and $$p_e(t) \underset{t\to t_2}{\longrightarrow} \frac{\gamma_+}{\gamma_+ + \gamma_-} = \frac{e^{-\hbar\omega_h\beta_h}}{1 + e^{-\hbar\omega_h\beta_h}}$$ **(since** $\frac{\gamma_+}{\gamma_-} = e^{-\hbar\omega_h\beta_h}$ **)**

Assuming that the duration of stroke 2 is sufficient for the qubit to thermalize with the hot bath, meaning $t_2 - t_1 \gg T_1 = (\gamma_+ + \gamma_-)^{-1}$

→ **Under such conditions, we therefore have**

$$p_e(t_2) = \frac{e^{-\hbar\omega_h\beta_h}}{1 + e^{-\hbar\omega_h\beta_h}}$$

I. Quantum Otto Engine

- **Evolution of the qubit state during stroke 2 (continuing):**

Similarly, for the coherences in the basis $\{|e\rangle, |g\rangle\}$, its dynamics is given by

$$\frac{d}{dt}c_{eg}(t) = \frac{d}{dt}\langle e|\rho_S(t)|g\rangle = \langle e|\frac{d}{dt}\rho_S(t)|g\rangle = \ldots = \Big[-i\omega_S - \frac{\gamma_+ + \gamma_-}{2} - \gamma_z\Big]c_{eg}(t)$$

(using the master equation of the previous page)

$$\Longrightarrow c_{eg}(t) = e^{-i\omega_s t}e^{-t/T_2}c_{eg}(0) \qquad \text{with} \quad T_2 := \left(\frac{\gamma_+ + \gamma_-}{2} + \gamma_z\right)^{-1}$$

Remembering that the stroke takes place between $[t_1;t_2]$, we replace 0 by t_1 and t by $t - t_1$ in the right-hand side, implying, for all t in $[t_1;t_2]$,

$$c_{eg}(t) = e^{-i\omega_h(t-t_1)}e^{-(t-t_1)/T_2}c_{eg}(t_1)$$

Then, since $\rho(t_1) = \rho^{\text{th}}(\beta_c, \omega_c)$ **(which is diagonal in the basis $\{|e\rangle, |g\rangle\}$),**

we have $c_{eg}(t_1) = 0 \;\Rightarrow c_{eg}(t) = 0 \;$ for all $t \in [t_1;t_2]$

Then, combining $c_{eg}(t_2) = 0$ and $p_e(t_2) = e^{-\hbar\omega_h\beta_h}/\left(1 + e^{-\hbar\omega_h\beta_h}\right)$ **we obtain:**

$$\rho(t_2) = p_e(t_2)|e\rangle\langle e| + [1 - p_e(t_2)]|g\rangle\langle g|$$

$$= \frac{e^{-\hbar\omega_h\beta_h}}{1 + e^{-\hbar\omega_h\beta_h}}|e\rangle\langle e| + \frac{1}{1 + e^{-\hbar\omega_h\beta_h}}|g\rangle\langle g| = \rho^{\text{th}}(\beta_h, \omega_h)$$

Thermal state at the hot bath's temperature

I. Quantum Otto Engine

- **Work exchange during stroke 2:**

$$W_2 := \int_{t_1}^{t_2} dt \mathrm{Tr}[\rho(t)\dot{H}(t)] \;= 0$$

since the Hamiltonian is time-independent during stroke 2

- **Heat exchange during stroke 1:**

$$\begin{aligned} Q_2 := \int_{t_1}^{t_2} dt \mathrm{Tr}[\dot{\rho}(t)H(t)] &= \int_{t_1}^{t_2} dt \mathrm{Tr}[\dot{\rho}(t)H] \\ &= \mathrm{Tr}\left\{\left[\int_{t_1}^{t_2} dt \dot{\rho}(t)\right] H\right\} \\ &= \mathrm{Tr}\left\{[\rho(t_2) - \rho(t_1)]H\right\} \\ &= \mathrm{Tr}\left\{\left[\rho^{\mathrm{th}}(\beta_h, \omega_h) - \rho^{\mathrm{th}}(\beta_c, \omega_c)\right] H\right\} \\ &= \hbar\omega_h \left[p_e(\beta_h, \omega_h) - p_e(\beta_c, \omega_c)\right] \end{aligned}$$

since the Hamiltonian is time-independent during stroke 2

$$\text{with} \quad p_e(\beta_h, \omega_h) := \frac{e^{-\hbar\omega_h\beta_h}}{1 + e^{-\hbar\omega_h\beta_h}}$$

→ **Similar calculation for stroke 4**

I. Quantum Otto Engine

- **Energetic balance over one cycle**

we find:

$$W_1 = \hbar(\omega_h - \omega_c)p_e(\beta_c, \omega_c) \qquad Q_1 = 0$$

$$W_2 = 0 \qquad Q_2 = \hbar\omega_h\Big[p_e(\beta_h, \omega_h) - p_e(\beta_c, \omega_c)\Big]$$

$$W_3 = \hbar(\omega_c - \omega_h)p_e(\beta_h, \omega_h) \qquad Q_3 = 0$$

$$W_4 = 0 \qquad Q_4 = \hbar\omega_c\Big[p_e(\beta_c, \omega_c) - p_e(\beta_h, \omega_h)\Big]$$

we can verify that $\underline{W_1 + Q_2 + W_3 + Q_4 = 0}$ as it should be since the the cycle is closed

- **Condition for work extraction**

❖ Work extracted per cycle (=the work exchange with the qubit over one cycle):

$$W_1 + W_2 + W_3 + W_4 = \hbar(\omega_h - \omega_c)[p_e(\beta_c, \omega_c) - p_e(\beta_h, \omega_h)]$$

❖ Over one cycle, we want to extract work, meaning we want $\underline{W_1 + W_2 + W_3 + W_4 \leq 0}$

$$\Longrightarrow \hbar(\omega_h - \omega_c)[p_e(\beta_c, \omega_c) - p_e(\beta_h, \omega_h)] \leq 0$$

$$\Longrightarrow p_e(\beta_c, \omega_c) \leq p_e(\beta_h, \omega_h)]$$

$$\Longrightarrow \frac{\omega_c}{\omega_h} \geq \frac{T_c}{T_h}$$

I. Quantum Otto Engine

- **Effciency**

As usual, the efficiency is defined as the ratio of the quantity we care about, here, the total extracted work $|W_1 + W_3| = -W_1 - W_3$, divided by its cost, here the heat provided by the hot bath, Q_2

$$\eta = \frac{|W_1 + W_3|}{Q_2} = \frac{-W_1 - W_3}{Q_2}$$

$$W_1 + Q_2 + W_3 + Q_4 = 0$$

$$= \frac{Q_2 + Q_4}{Q_2}$$

$$Q_4 = \hbar\omega_c\Big[p_e(\beta_c,\omega_c) - p_e(\beta_h,\omega_h)\Big]$$
$$Q_2 = \hbar\omega_h\Big[p_e(\beta_h,\omega_h) - p_e(\beta_c,\omega_c)\Big]$$

$$= 1 + \frac{\hbar\omega_c\Big[p_e(\beta_c,\omega_c) - p_e(\beta_h,\omega_h)\Big]}{\hbar\omega_h\Big[p_e(\beta_h,\omega_h) - p_e(\beta_c,\omega_c)\Big]} = 1 - \frac{\omega_c}{\omega_h} \longrightarrow \boxed{\eta = 1 - \frac{\omega_c}{\omega_h}}$$

called the Otto efficiency

- **Observation:** ❖ We found the same Otto efficiency as for classical Otto cycle

❖ We know that for classical heat engines, their efficiency is always upper bounded by the Carnot efficiency $\eta_{Carnot} = 1 - \frac{T_c}{T_h}$

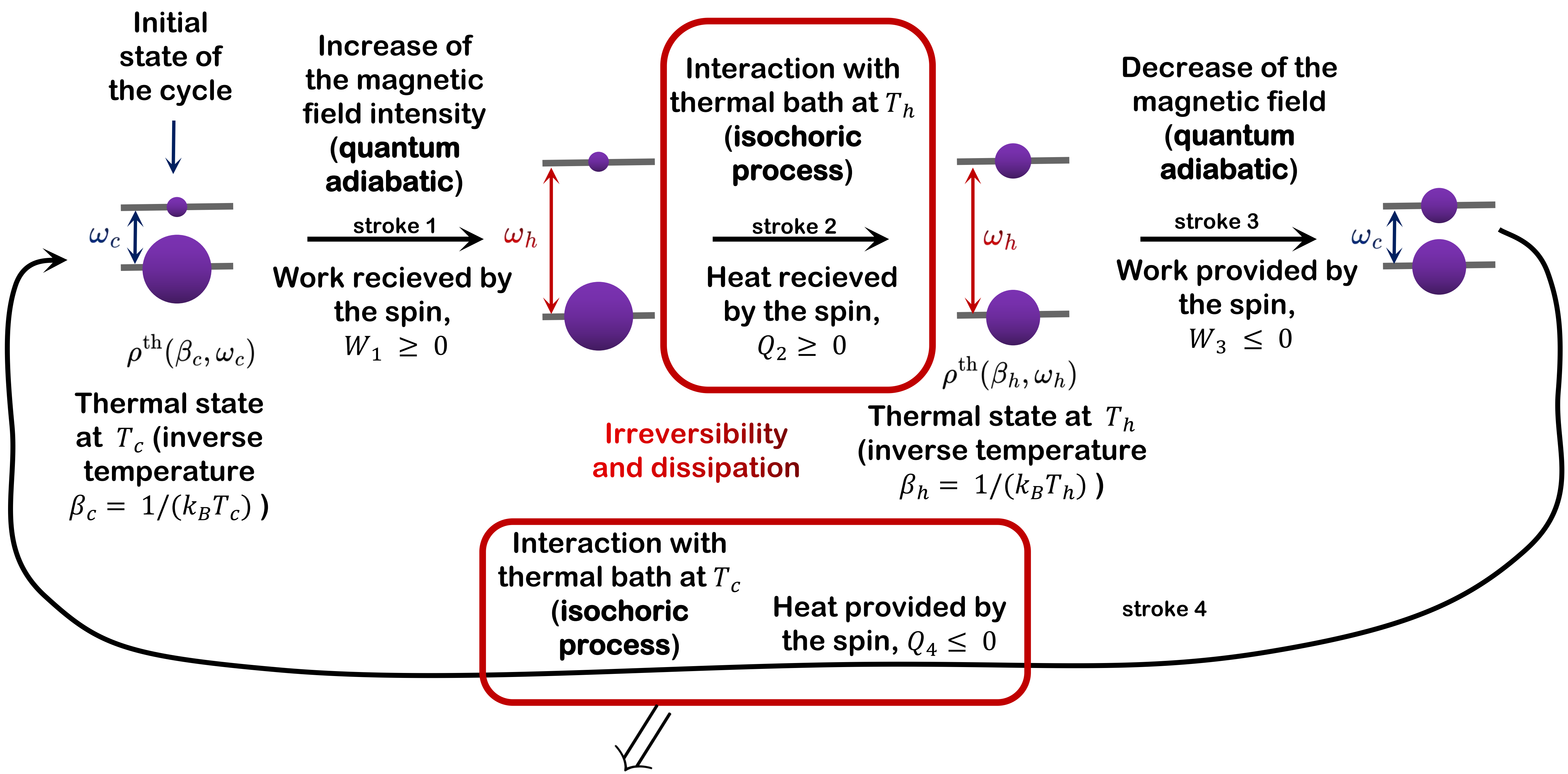
$\longrightarrow$ here $\eta = 1 - \frac{\omega_c}{\omega_h} \leq 1 - \frac{T_c}{T_h} = \eta_{\mathrm{Carnot}} \Longrightarrow$ here as well we are upper bounded by the Carnot effciciency!

work extraction condition $\frac{\omega_c}{\omega_h} \geq \frac{T_c}{T_h}$

- **Question: Why don't we reach the Carnot efficiency?**

I. Quantum Otto Engine

- **Reason for not reaching the Carnot efficiency**



Strokes 2 and 4, in contact with the hot bath and cold bath are irrevesible!

This is because at the moment of the interaction with the hot bath, the qubit is not in equilibrium with the hot bath (meaning the qubit is not in the thermal state at temperature T_h); same thing for the cold bath, at the moment of the interaction, the qubit is not in equilibrium with the cold bath.
→ This leads to irreversibility (expressed by a striclty positive entropy production), loss of extractable work, and therefore loss of efficiency

$\Longrightarrow$ **Loss of extractable work**

$\Downarrow$

Loss of efficiency

II. Irreversibility, entropy production, and second law

II. Irreversibility, entropy production, and second law

A. General definition and context

Let's consider a driven quantum system S interacting with a bath B in a thermal state at temperature T:

$$H_{SB}(t) = H_S(t) + V_{SB} + H_B$$

We assume no initial correlations between system and bath:

$$\rho_{SB}(0) = \rho_S(0) \otimes \rho_B^{\text{th}}(\beta)$$

$$= Z_B^{-1} e^{-\beta H_B}$$

$$\beta := (k_B T)^{-1}$$

$$Z_B := \text{Tr}\left[e^{-\beta H_B}\right]$$

The **entropy production** at time t associated with the **open evolution/dynamics** of the quantum system S is:

$$\Sigma_S(t) := \Delta S_S - \beta Q_S(t)$$

(M. Esposito, K. Lindenberg, C. Van den Broeck, New J Phys 12, 013013 (2010))

with

$$\begin{cases} \Delta S_S = S_{\text{vN}}[\rho_S(t)] - S_{\text{vN}}[\rho_S(0)] \\ Q_S(t) := -\Delta E_B = -\text{Tr}\{\rho_B(t) H_B\} + \text{Tr}\{\underbrace{\rho_B(0)}_{\rho_B^{\text{th}}(\beta)} H_B\} \end{cases}$$

II. Irreversibility, entropy production, and second law

Entropy production for the quantum system S

$$\Sigma_S(t) := \Delta S_S - \beta Q_S(t)$$

Observations:

- $\Sigma(t)$ has the same form as in classical thermodynamics
- $\Sigma(t) \geq 0$ (always positive when there is no initial correlations between system and bath, meaning when the intial state is separable: $\rho_{SB}(0) = \rho_S(0) \otimes \rho_B^{\mathrm{th}}(\beta)$)
- **Second law of thermodynamics in the quantum regime**

$$\Delta S_S = \underbrace{S_{\mathrm{flow}}(t)}_{\substack{= \beta Q \\ = -\beta \Delta E_B}} + \underbrace{\Sigma(t)}_{\geq 0}$$

Term of exchange of entropy between system and bath = heat flow

II. Irreversibility, entropy production, and second law

Entropy production for the quantum system S

$$\Sigma_S(t) := \Delta S_S - \beta Q_S(t)$$

Informational form: The entropy production can be re-expressed as

$$\Sigma_S(t) = I_{S:B}(t) + D[\rho_B(t)|\rho_B^{\mathrm{th}}(\beta)]$$

Represents the generation of **correlations** between S and B

Quantifies how much the state of the bath changed from 0 to t

$$D[\rho_B(t)|\rho_B^{\mathrm{th}}(T)] := \mathrm{Tr}\{\rho_B(t)[\ln \rho_B(t) - \ln \rho_B^{\mathrm{th}}(T)]\}$$

relative entropy = “pseudo distance”

$$I_{S:B}(t) := S_{\mathrm{vN}}[\rho_S(t)] + S_{\mathrm{vN}}[\rho_B(t)] - S_{\mathrm{vN}}[\rho_{SB}(t)]$$

mutual information = measure of the correlation between S and B

In this form, $\Sigma(t)$ appears to represents the amount of information lost in the bath

Irreversibility, at the quantum scale, comes from this loss of information

II. Irreversibility, entropy production, and second law

Observations:

- This definition of entropy production is valid for S and B initially uncorrelated (meaning $\rho_{SB}(0) = \rho_A \otimes \rho_B(0)$), with B being initially in a thermal state, and for **arbitrary coupling strength** between S and B.

- In the limit of weak coupling, we saw (see Lecture 1) that all definitions of heat become equivalent to the weak coupling expression:

$$Q_S(t) = \int_0^t du \mathrm{Tr}\{\dot{\rho}_S(u) H_S(u)\}$$

- Then, in the weak coupling limit, the entropy production can be expressed with quantities of S only:

$$\Sigma_S(t) = \Delta S_S - \beta Q_S(t)$$

$$= \int_0^t du \mathrm{Tr}\{\dot{\rho}_S(u) H_S(u)\}$$

B. Entropy production at the quantum trajectory level

- We consider a system S interacting with a bath B from a time t_0 to t_1
- The resulting unitary operation on SB is denoted $U_{SB}(t_0, t_1)$ or simply U_{SB}
- Uncorrelated initial state: $\rho_{SB} = \rho_S \otimes \rho_B$
- We express the local state of S and B in their repective eigenbasis:

$$\rho_S = \sum_n p_n |n\rangle\langle n| \qquad\qquad \rho_B = \sum_\nu q_\nu |\nu\rangle\langle\nu|$$

- The final (average) state of SB is: $\rho'_{SB} = U_{SB}\rho_{SB}U^\dagger_{SB}$
- The reduced final (average) state of S is: $\rho'_S := \mathrm{Tr}_B[\rho'_{SB}]$
- The reduced final (average) state of B is: $\rho'_B := \mathrm{Tr}_S[\rho'_{SB}]$
- We can also express the local final (average) states in their respective eigenbasis:

$$\rho'_S = \sum_m p'_m |\psi_m\rangle\langle\psi_m| \qquad\qquad \rho'_B = \sum_\mu q'_\mu |\phi_\mu\rangle\langle\phi_\mu|$$

Reminder: Quantum measurements - projective measurements

→ A projective measurement is described by a collection of projectors:

$$\{P_r\}_r \quad \text{with} \quad \begin{cases} \sum_r P_r = \mathbb{I} \\ P_r \text{ is a projector, } P_r^2 = P_r \\ r\text{: outcome of the measurement} \end{cases}$$

→ measurement: $$\rho \xrightarrow{\text{outcome r}} \rho_{|r} = \frac{1}{p_r} P_r \rho P_r$$

probability of outcome r: $$p_r = \mathrm{Tr}[\rho P_r]$$

❖ Typical situation: we have a basis $\{|\varphi_r\rangle\}_r$ (often the eigenbasis of an observable)

→ $P_r = |\varphi_r\rangle\langle\varphi_r|$ → $\rho_{|r} = |\varphi_r\rangle\langle\varphi_r|$

→ $p_r = \mathrm{Tr}[\rho P_r] = \langle\varphi_r|\rho|\varphi_r\rangle$

II. Irreversibility, entropy production, and second law

We consider the following experimental protocol - the **forward** process

- (i) Measure of the state of S: $\rightarrow$ Measure S in the basis $\{|n\rangle\}_n$ (= eigenbasis of ρ_S)

  ⟶ we obtain the result n with probability $p_n = \langle n|\rho_S|n\rangle$

  ⟶ state of S just after the measure: $|n\rangle\langle n|$

- (ii) Measure of the state of B: $\rightarrow$ Measure B in the basis $\{|\nu\rangle\}_\nu$ (= eigenbasis of ρ_B)

  ⟶ we obtain the result ν with probability $q_\nu = \langle \nu|\rho_B|\nu\rangle$

  ⟶ state of B just after the measure: $|\nu\rangle\langle\nu|$

- (iii) Unitary evolution U_{SB}

  ⟶ state of SB after the evolution: $U_{SB}|n\rangle\langle n|\otimes|\nu\rangle\langle\nu|U_{SB}^\dagger$

II. Irreversibility, entropy production, and second law

- (iv) Measure of SB in the basis $\{|\psi_m\rangle\otimes|\phi_\mu\rangle\}_{m,\mu}$ (= eigenbasis of ρ'_{SB})

  → we obtain the result m, μ with probability:

$$\langle\psi_m|\langle\phi_\mu|\Big[U|n\rangle|\nu\rangle\langle n|\langle\nu|U^\dagger\Big]|\psi_m\rangle|\phi_\mu\rangle = \langle\psi_m|\langle\phi_\mu|U|n\rangle|\nu\rangle\langle n|\langle\nu|U^\dagger|\psi_m\rangle|\phi_\mu\rangle$$

$$= \left|\langle\psi_m|\langle\phi_\mu|U|n\rangle|\nu\rangle\right|^2$$

  → state of SB just after the measure: $|\psi_m\rangle\langle\psi_m|\otimes|\phi_\mu\rangle\langle\phi_\mu|$

→ We obtain the "trajectory" $n,\nu \;\rightarrow\; m,\mu$ with a probability:

$$P_F(n,\nu,m,\mu) = p_n q_\nu \left|\langle\psi_m|\langle\phi_\mu|U|n\rangle|\nu\rangle\right|^2$$

→ Stochastic process: each quantum trajectory $n,\nu \;\rightarrow\; m,\mu$ happens with a probability $P_F(n,\nu\,,m,\mu)$

II. Irreversibility, entropy production, and second law

We consider the following **backward** experimetnal process

- ❖ The aim is to realize the backward process
- ❖ But we will consider situations in which we **lost some information during the forward process**
- ❖ For that, we will start the backward process from a state $\widetilde{\rho_{SB}}$ which is slightly different from the final state of the forward process, ρ'_{SB}

- (i) Measure of SB in the basis $\{|\psi_m\rangle \otimes |\phi_\mu\rangle\}_{m,\mu}$ (= eigenbasis of ρ'_{SB})

⟶ we obtain the result m, μ with probability:

$$\langle\psi_m|\langle\phi_\mu|\tilde{\rho}_{SB}|\psi_m\rangle|\phi_\mu\rangle$$

⟶ state of SB just after the measure: $|\psi_m\rangle\langle\psi_m| \otimes |\phi_\mu\rangle\langle\phi_\mu|$

- (ii) Reverse unitary evolution: $U_{SB}^\dagger$

⟶ state of SB after the evolution: $U_{SB}^\dagger|\psi_m\rangle\langle\psi_m| \otimes |\phi_\mu\rangle\langle\phi_\mu| U_{SB}$

II. Irreversibility, entropy production, and second law

- (iii) Measure of SB in the basis $\{|n\rangle\otimes|\nu\rangle\}_{n,\nu}$ (= eigenbasis of ρ_{SB})

  ⟶ we obtain the result n, ν with probability:

$$\langle n|\langle\nu|\left[U_{SB}^{\dagger}|\psi_m\rangle|\phi_\mu\rangle\langle\psi_m|\langle\phi_\mu|U_{SB}\right]|n\rangle|\nu\rangle = \underbrace{\langle n|\langle\nu|U_{SB}^{\dagger}|\psi_m\rangle|\phi_\mu\rangle}\,\underbrace{\langle\psi_m|\langle\phi_\mu|U_{SB}|n\rangle|\nu\rangle}$$

$$= \left|\langle n|\langle\nu|U_{SB}^{\dagger}|\psi_m\rangle|\phi_\mu\rangle\right|^2$$

  ⟶ state of SB just after the measure: $|n\rangle\langle n|\otimes|\nu\rangle\langle\nu|$

→ We obtain the "trajectory" $m,\mu \ \rightarrow \ n,\nu$ with a probability:

$$P_B(n,\nu,m,\mu) = \langle\psi_m|\langle\phi_\mu|\tilde{\rho}_{SB}|\psi_m\rangle|\phi_\mu\rangle \times \left|\langle n|\langle\nu|U_{SB}^{\dagger}|\psi_m\rangle|\phi_\mu\rangle\right|^2$$

→ Stochastic process: each quantum trajectory $m,\mu \ \rightarrow \ n,\nu$ happens with a probability $P_B(n,\nu\,,m,\mu)$

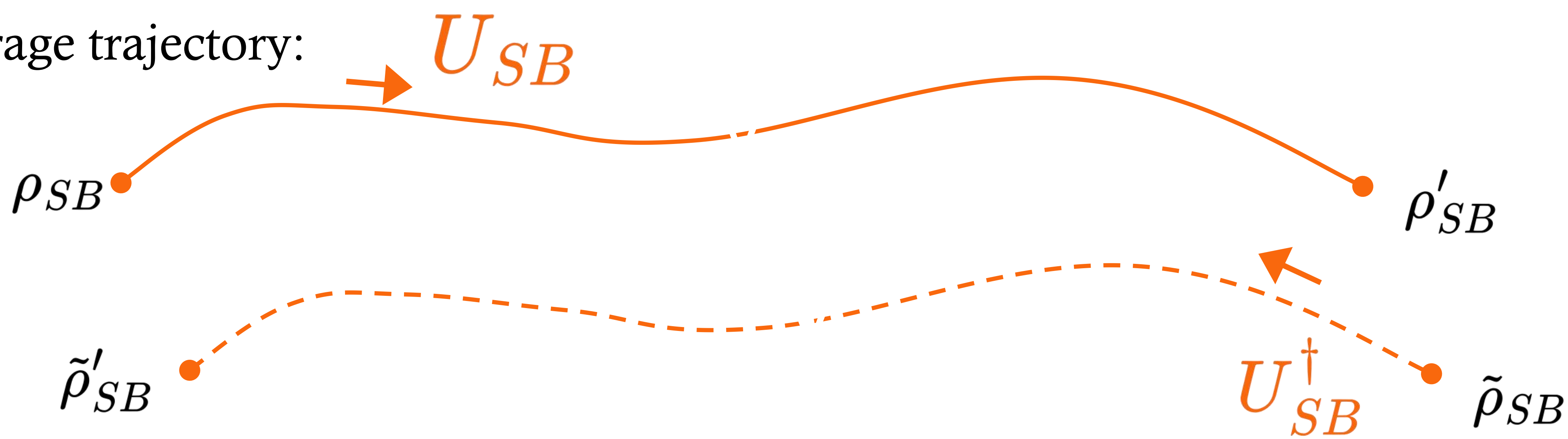
II. Irreversibility, entropy production, and second law

- Stochastic entropy production (for one given trajectory):

$$\sigma_{n,\nu,m\mu} := \ln \frac{P_F(n,\nu,m,\mu)}{P_B(n,\nu,m,\mu)} = \ln \frac{p_n q_\nu}{\langle\psi_m|\langle\phi_\mu|\tilde{\rho}_{SB}|\psi_m\rangle|\phi_\mu\rangle}$$

The entropy production (for one trajectory) is determined by how unlikely the backward process is compared to the forward process

- Average trajectory:



- Average entropy production: $\langle\sigma\rangle := \sum_{n,\nu,m,\mu} P_F(n,\nu,m,\mu)\sigma_{n,\nu,m,\mu}$

$$= \sum_{n,\nu,m\mu} P_F(n,\nu,m,\mu) \ln \frac{p_n q_\nu}{\langle\psi_m|\langle\phi_\mu|\tilde{\rho}_{SB}|\psi_m\rangle|\phi_\mu\rangle}$$

$$= \sum_{n,\nu,m\mu} p_n q_\nu |\langle\psi_m|\langle\phi_\mu|U|n\rangle|\nu\rangle|^2 \ln \frac{p_n q_\nu}{\langle\psi_m|\langle\phi_\mu|\tilde{\rho}_{SB}|\psi_m\rangle|\phi_\mu\rangle}$$

II. Irreversibility, entropy production, and second law

Average entropy production for different choice of $\widetilde{\rho_{SB}}$ (the initial state of the backward process)

- (a) We assume we lost the information of the final correlations between S and B (for the forward process), and the change of state of B:

$$\tilde{\rho}_{AB} = \rho'_S \otimes \rho_B$$

$\longrightarrow$ Doing the math (***exercise***), we obtain $\langle \sigma \rangle = \underbrace{I_{S:B}(t_1) + D[\rho'_B | \rho_B]}_{\text{precisely the information we lost!!}}$

- (b) Now we assume we only lost the information of the final (for the forward process) correlations between S and B: $\tilde{\rho}_{AB} = \rho'_S \otimes \rho'_B$

$\longrightarrow$ Doing the math (***exercise***), we obtain $\underbrace{\langle \sigma \rangle = I_{S:B}(t_1)}_{\text{precisely the information we lost!!}}$

- (c) Now we assume we did not lost any information:

$\tilde{\rho}_{AB} = \rho'_{SB}$ $\longrightarrow$ Doing the math (***exercise***), we obtain $\langle \sigma \rangle = 0$!!

II. Irreversibility, entropy production, and second law

Conclusion:

- The entropy production indeed represents the **lost information**
- This lost information prevents ones to exaclty perform the reverse process
- This loss of information is responsible for the **irreversibility** of the dynamics

Observation: This also show the subjective character of entropy production: it depends on the level of accessible information

References:

G. Manzano, J. M. Horowitz, J. M. R. Parrondo: Quantum Fluctuation Theorems for Arbitrary Environments: Adiabatic and Nonadiabatic Entropy Production. Phys. Rev. X 8, 031037 (2018)

G. T. Landi, M. Paternostro: Irreversible entropy production: From classical to quantum. Rev. Mod. Phys. 93, 035008 (2021)

II. Irreversibility, entropy production, and second law

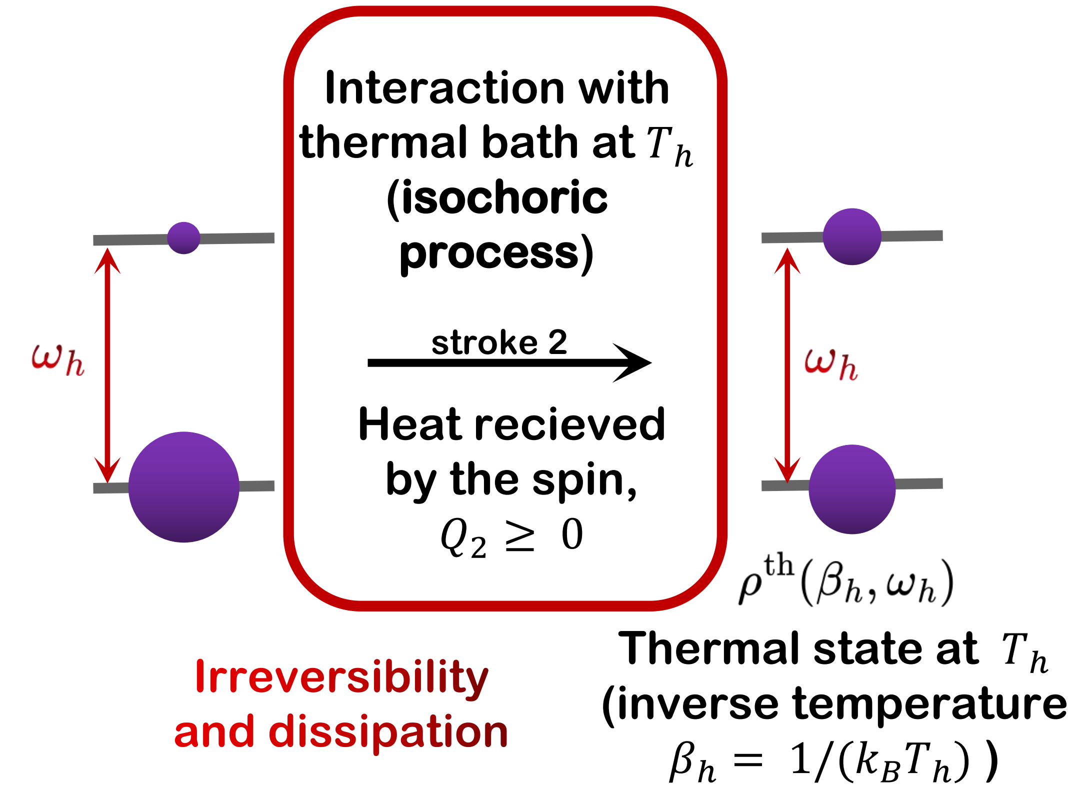
Entropy production is a central quantity in

- Efficiency of Quantum Heat engine and quantum refrigerator
- Energy cost of quantum reset (extension of Landauer principle)
- Energy cost of noisy quantum operation
- Fluctuations (Thermodynamic Uncertainty Relation, TUR)
- Accuracy of Quantum Clock

II. Irreversibility, entropy production, and second law

C. Application: calcul of the entropy production during an Otto cycle

(i) Entropy production during stroke 2



$$\longrightarrow \quad \Sigma^{(2)} = \underbrace{\Delta S_S} - \beta_h \underbrace{Q_S(t_2)}$$

variation of entropy of the qubit S during stroke 2

$$= S[\rho_S(t_2)] - S[\rho_S(t_1)]$$

$$= S[\rho^{\text{th}}(\beta_h, \omega_h)] - S[\rho^{\text{th}}(\beta_c, \omega_c)]$$

heat exchange during stroke 2

$$= Q_2 = \hbar\omega_h \Big[p_e(\beta_h, \omega_h) - p_e(\beta_c, \omega_c) \Big]$$

II. Irreversibility, entropy production, and second law

- von Neumann entropy of the thermal states:

$$S[\rho^{\rm th}(\beta_h,\omega_h)] = -\mathrm{Tr}[\rho^{\rm th}(\beta_h,\omega_h)\ln\rho^{\rm th}(\beta_h,\omega_h)]$$

$$\rho^{\rm th}(\beta_h,\omega_h) = \frac{e^{-\hbar\omega_h\beta_h}}{1+e^{-\hbar\omega_h\beta_h}}|e\rangle\langle e| + \frac{1}{1+e^{-\hbar\omega_h\beta_h}}|g\rangle\langle g|$$

$$= p_e(\beta_h,\omega_h)|e\rangle\langle e| + (1-p_e(\beta_h,\omega_h)|g\rangle\langle g|$$

$$p_e(\beta_h,\omega_h) = \frac{e^{-\hbar\omega_h\beta_h}}{1+e^{-\hbar\omega_h\beta_h}}$$

$$\implies S[\rho^{\rm th}(\beta_h,\omega_h)] = -p_e(\beta_h,\omega_h)\underbrace{\ln p_e(\beta_h,\omega_h)}_{\substack{=\ln\frac{e^{-\hbar\omega_h\beta_h}}{1+e^{-\hbar\omega_h\beta_h}} \\ = -\hbar\omega_h\beta_h - \ln(1+e^{-\hbar\omega_h\beta_h})}} - [1-p_e(\beta_h,\omega_h)]\underbrace{\ln[1-p_e(\beta_h,\omega_h)]}_{\substack{=\ln\frac{1}{1+e^{-\hbar\omega_h\beta_h}} \\ = -\ln(1+e^{-\hbar\omega_h\beta_h})}}$$

$$\implies S[\rho^{\rm th}(\beta_h,\omega_h)] = \hbar\omega_h\beta_h p_e(\beta_h,\omega_h) + \ln(1+e^{-\hbar\omega_h\beta_h})$$

$\longrightarrow$ Similarly, $S[\rho^{\rm th}(\beta_c,\omega_c)] = \hbar\omega_c\beta_c p_e(\beta_c,\omega_c) + \ln(1+e^{-\hbar\omega_c\beta_c})$

II. Irreversibility, entropy production, and second law

- All together, the entropy production during stroke 2 is:

$$\Sigma^{(2)} = S[\rho^{\text{th}}(\beta_h,\omega_h)] - S[\rho^{\text{th}}(\beta_c,\omega_c)] - \hbar\omega_h\beta_h\Big[p_e(\beta_h,\omega_h) - p_e(\beta_c,\omega_c)\Big]$$

$$= \hbar\omega_h\beta_h p_e(\beta_h,\omega_h) + \ln(1+e^{-\hbar\omega_h\beta_h}) - \hbar\omega_c\beta_c p_e(\beta_c,\omega_c) - \ln(1+e^{-\hbar\omega_c\beta_c}) - \hbar\omega_h\beta_h\Big[p_e(\beta_h,\omega_h) - p_e(\beta_c,\omega_c)\Big]$$

$$= \hbar(\omega_h\beta_h - \omega_c\beta_c)p_e(\beta_c,\omega_c) + \ln(1+e^{-\hbar\omega_h\beta_h}) - \ln(1+e^{-\hbar\omega_c\beta_c})$$

→ The entropy production during stroke 2 is indeed due to the fact that the state of S just before the interaction with the hot bath is $\rho_S(t_1) = \rho^{th}(\beta_c,\omega_c)$ which is different from the hot equilibrium state $\rho^{th}(\beta_h,\omega_h)$

- Similarly, the entropy production during stroke 4 is:

$$\Sigma^{(4)} = S[\rho^{\text{th}}(\beta_c,\omega_c)] - S[\rho^{\text{th}}(\beta_h,\omega_h)] - \beta_c \underbrace{Q_4}_{=\hbar\omega_c\left[p_e(\beta_c,\omega_c) - p_e(\beta_h,\omega_h)\right]}$$

$$= \hbar(\omega_c\beta_c - \omega_h\beta_h)p_e(\beta_h,\omega_h) + \ln(1+e^{-\hbar\omega_c\beta_c}) - \ln(1+e^{-\hbar\omega_h\beta_h})$$

→ The entropy production during stroke 4 is indeed due to the fact that the state of S just before the interaction with the cold bath is $\rho_S(t_3) = \rho^{th}(\beta_h,\omega_h)$ which is different from the cold equilibrium state $\rho^{th}(\beta_c,\omega_c)$

II. Irreversibility, entropy production, and second law

- All together, the entropy production during a whole Otto cycle is

$$\Sigma^{\text{Otto}} = \Sigma^{(2)} + \Sigma^{(4)}$$

$$= \hbar(\omega_h \beta_h - \omega_c \beta_c)\Big[p_e(\beta_c, \omega_c) - p_e(\beta_h, \omega_h)\Big]$$

- Additionally, one can show explicitly that $\Sigma^{(2)}$ and $\Sigma^{(4)}$ are indeed related to loss of extractable work; more precisely

$$\underbrace{-W_1 - W_3}_{\text{work extracted during one Otto cycle}} = \text{maximal extractable work} - \frac{\Sigma^{(2)}}{\beta_h} - \frac{\Sigma^{(4)}}{\beta_c}$$

$$= \text{work extracted during one Carnot cycle} - \frac{\Sigma^{(2)}}{\beta_h} - \frac{\Sigma^{(4)}}{\beta_c}$$

(see more detail in next section)

III. Quantum Carnot Engine

III. Quantum Carnot Engine

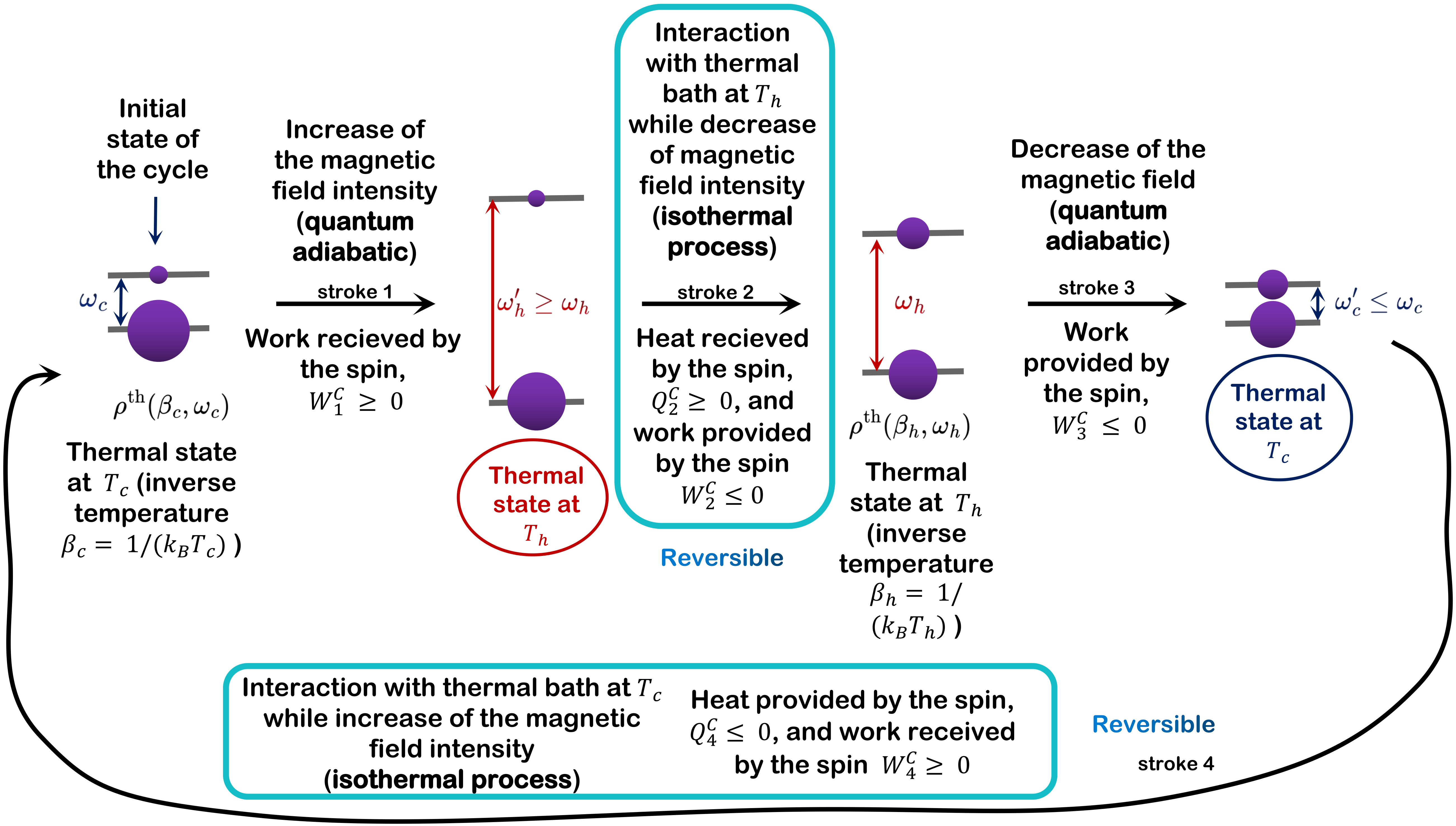
Quantum Carnot cycle with a qubit (spin-1/2)

Observation: as for the Otto cycle, we define the ground state energy of the spin to be equal to 0

$$\longrightarrow \quad H = \hbar\omega|e\rangle\langle e| \quad \text{with, for a spin-1/2,} \quad \omega = \gamma|\vec{B}|$$

(in the following, $\hbar = 1$)

→ the energy splitting is controled by the intensity of the applied magnetic field



III. Quantum Carnot Engine

- **The cyclic structure imposes**

$$W_1^C + W_2^C + Q_2^C + W_3^C + W_4^C + Q_4^C = 0$$

- **Efficiency**

Carnot efficiency

$$\eta_{\mathrm{Carnot}} = \frac{-(W_1^C + W_2^C + W_3^C + W_4^C)}{Q_2^C} = \ldots = \boxed{1 - \frac{T_c}{T_h}}$$

See following additional details

- **The cycle is reversible and there is no entropy production since the state of S is in thermal equilibrium before each interaction with the respective thermal bath (stroke 2 and 4)**

III. Quantum Carnot Engine

Additional details (exercise)

- In order to obtain a thermal state at temperature T_h at the end of the first stroke, one can show that we must have
$$\omega_h' = \omega_c \frac{T_h}{T_c}$$
- Similarly, in order to obtain a thermal state at temperature T_c at the end of the third stroke, one can show that we must have
$$\omega_c' = \omega_h \frac{T_c}{T_h}$$
- Stroke 1: we can show: $W_1^C = \hbar(\omega_h' - \omega_c) p_e(\beta_c, \omega_c)$ and $Q_1^C = 0$
- Stroke 2: assuming quasi-static decrease of the amplitude of the magnetic field (so that the qubits remains at all time in the instantaneous thermal state), we can show:
$$W_2^C = T_h \ln\left(1 + e^{-\hbar\omega_c\beta_c}\right) - T_h \ln\left(1 + e^{-\hbar\omega_h\beta_h}\right) \text{ and } Q_2^C = \hbar\omega_h p_e(\beta_h, \omega_h) - \hbar\omega_h' p_e(\beta_c, \omega_c) - W_2^C$$
- Stroke 3: we can show: $W_3^C = \hbar(\omega_c' - \omega_h) p_e(\beta_h, \omega_h)$ and $Q_3^C = 0$
- Stroke 4: assuming quasi-static increase of the amplitude of the magnetic field (so that the qubits remains at all time in the instantaneous thermal state), we can show:
$$W_4^C = T_c \ln\left(1 + e^{-\hbar\omega_h\beta_h}\right) - T_c \ln\left(1 + e^{-\hbar\omega_c\beta_c}\right) \text{ and } Q_4^C = \hbar\omega_c p_e(\beta_c, \omega_c) - \hbar\omega_c' p_e(\beta_h, \omega_h) - W_4^C$$
- With all that, we obtain
$$\eta_{\text{Carnot}} = \frac{-(W_1^C + W_2^C + W_3^C + W_4^C)}{Q_2^C} = 1 - \frac{T_c}{T_h}$$
- We can also show that the extracted work over one Carnot cycle is maximal and equal to the work extracted over one Otto cycle plus the dissipated work during the irreversible strokes of the Otto cycle:
$$\text{work extracted during one Carnot cycle} = \text{ work extracted during one Otto cycle} + \frac{\Sigma^{(2)}}{\beta_h} + \frac{\Sigma^{(4)}}{\beta_c}$$

IV. The question of quantum advantages

IV. Quantum advantages in quantum engines

- **Bath of harmonic oscillator in coherent or squeezed states**

  → **Results in increase of efficiency**

  → **Recently realized experimentally** J. Klaers, S. Faelt, A. Imamoglu, E. Togan: Squeezed Thermal Reservoirs as a Resource for a Nanomechanical Engine beyond the Carnot Limit. Phys. Rev. X 7, 031044 (2017)

- **Bath containing quantum coherences (atoms prepared with quantum coherences between some energy levels)**

  → **Results in increase of efficiency**

  → **Recently realized experimentally** J. Kim, S.-h. Oh, D. Yang, J. Kim, M. Lee et al.: A photonic quantum engine driven by superradiance. Nat. Photonics 16, 707 (2022)

- **Bath containing quantum correlations (more precisely, bath composed of pairs of entangled atoms)**

  → **Results in increase of efficiency**

- **Collective effects: instead of having a single qubit undergoing the cycle, we use an ensemble of qubits. Interesting collective effects can emerge, akin to superradiance.**

  → **Results in increase of output power**

  → **Recently realized experimentally** J. Kim, S.-h. Oh, D. Yang, J. Kim, M. Lee et al.: A photonic quantum engine driven by superradiance. Nat. Photonics 16, 707 (2022)

- **Strong coupling effects: strong coupling between the system and the baths**

  → **Results in increase of output power**

IV. Quantum advantages in quantum engines

- <u>Observation</u>: in the above examples, one can outperform classical thermal engines (and refrigerators), ***but we use additional resources*** (quantum coherences, quantum correlations, entanglement, squeezing). Then, in some sense, we do more but with more resources. **<u>Can we do more with strictly the same resources?</u>**

Yes !

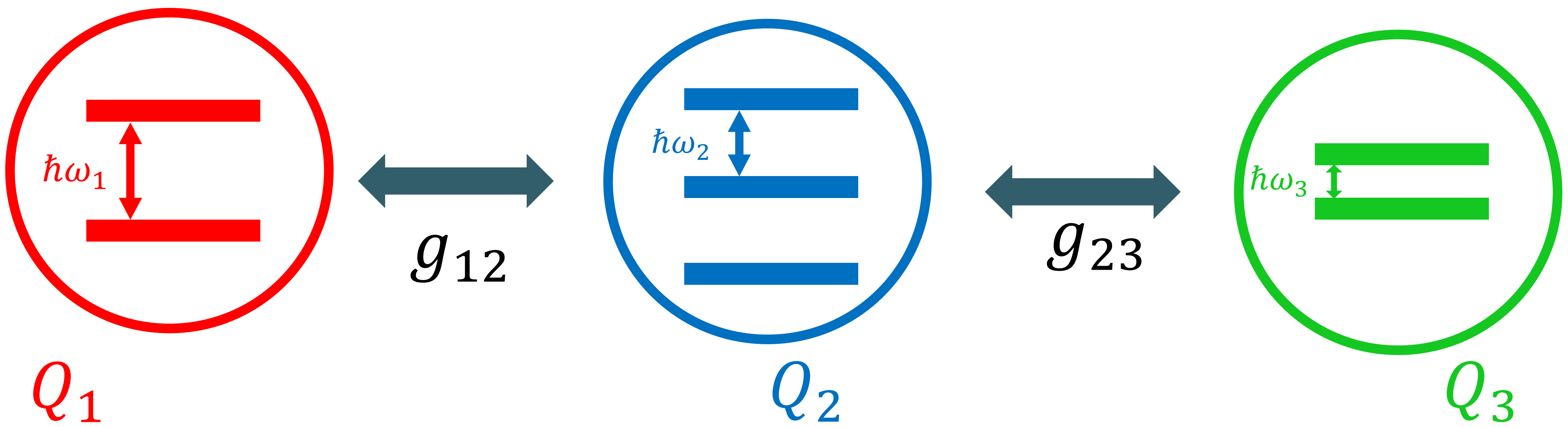
- **Collective effects, in some sense, provide such quantum advantages**
- **Quantum coherences generated during the dynamics can induce such genuine quantum advantage in term of fluctuations (the quantum engine is more stable, meaning it exhibits smaller fluctuations compared to its classical counterpart).**

J. A. Almanza-Marrero, G. Manzano: Reassessing quantum-thermodynamic enhancements in continuous thermal machines. arXiv:2403.19280 (2024)

V. Experimental implementation of quantum autonomous refrigerator

V. Experimental implementation

Autonomous reset of a qubit



Target qubit

Resonance condition

$$\omega_1 + \omega_3 = 2\omega_2$$

$$\Downarrow$$

Effective three body interaction:

$$g\Big(\underbrace{|101\rangle\langle 020|}_{\text{Remove 1 excitation from } Q_3 \text{ and } Q_1\text{, while adding 2 excitations to } Q_2} + \underbrace{|020\rangle\langle 101|}_{\text{The other way around}}\Big)$$

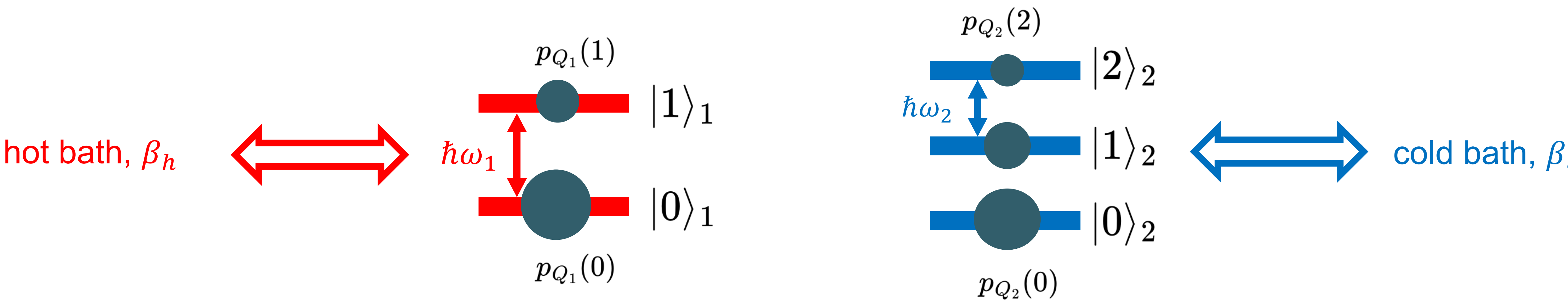
with

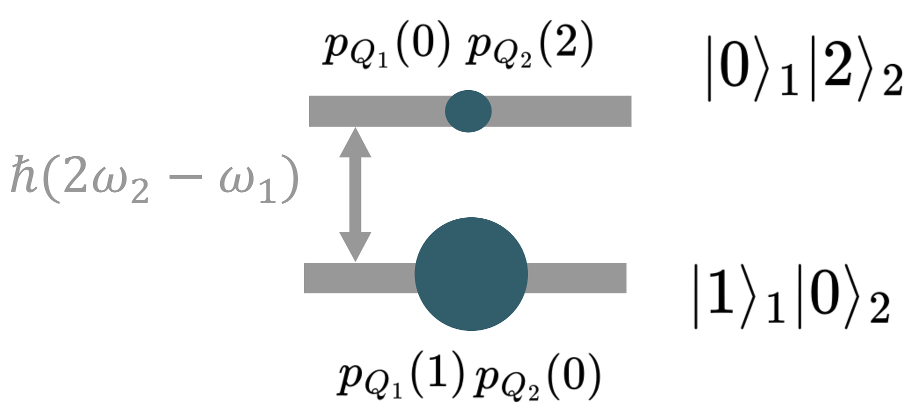
$$|ijk\rangle := |i\rangle_1 |j\rangle_2 |k\rangle_3$$

- How to ensure that only the transition removing 1 excitation from Q_1 occurs?

→ Symmetry breaking with temperature gradient:

$$\rho_{Q_1}(0) = e^{-\beta_h H_{Q_1}} / \mathrm{Tr}[e^{-\beta_h H_{Q_1}}] \qquad \rho_{Q_2}(0) = e^{-\beta_c H_{Q_2}} / \mathrm{Tr}[e^{-\beta_c H_{Q_2}}]$$





→ Effective temperature β_{eff}:

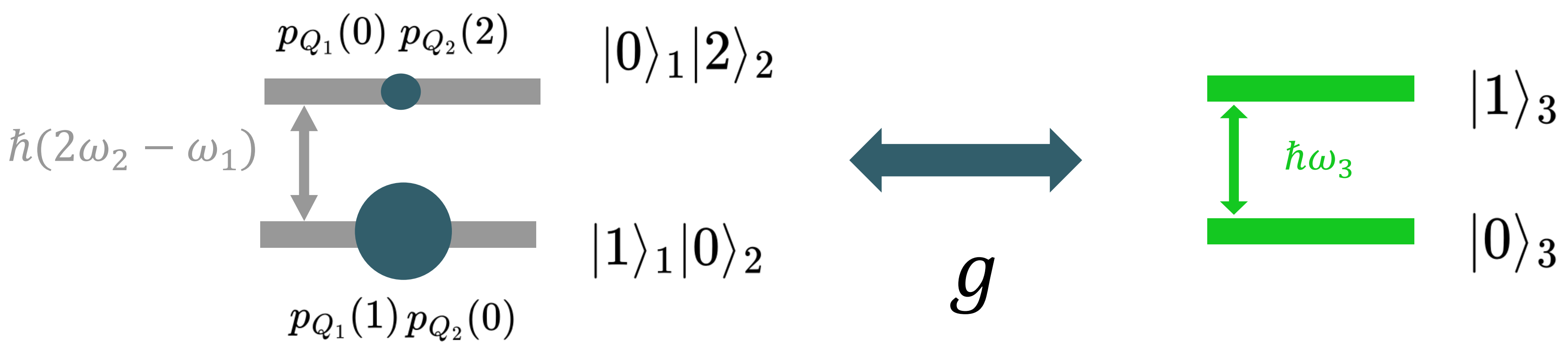
$$e^{-\beta_{\mathrm{eff}}(2\omega_2 - \omega_1)} = \frac{p_{Q_1}(0)\, p_{Q_2}(2)}{p_{Q_1}(1)\, p_{Q_2}(0)}$$

→ $$T_{eff} = \beta_{eff}^{-1} = T_c \frac{2\omega_2 - \omega_1}{2\omega_2 - \omega_1 T_c / T_h}$$

V. Experimental implementation

- Conclusion:

The transition $|1\rangle_3 \leftrightarrow |0\rangle_3$ is resonant and coupled with the transition $|0\rangle_1|2\rangle_2 \leftrightarrow |1\rangle_1\,|0\rangle_2$



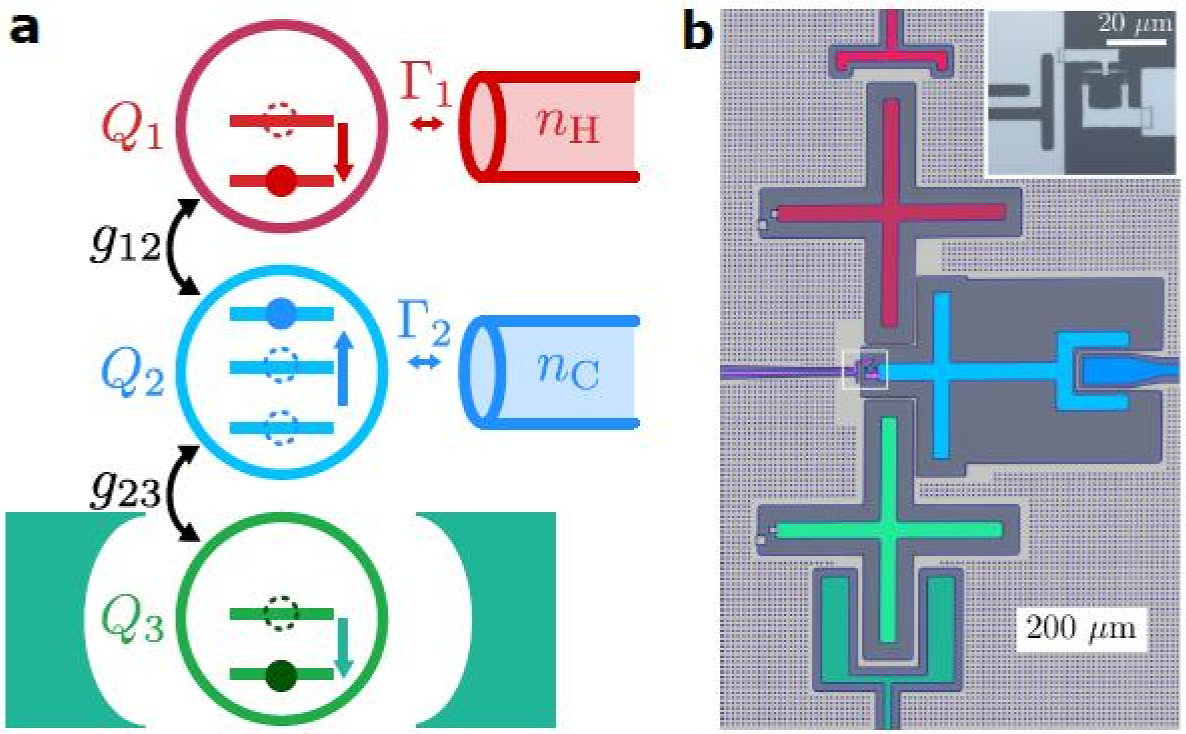
$$T_{eff} = \beta_{eff}^{-1} = T_c \frac{2\omega_2 - \omega_1}{2\omega_2 - \omega_1 T_c/T_h}$$

→ The transition $|1\rangle_3 \leftrightarrow |0\rangle_3$ will thermalize to the effective temperature T_{eff}

→ Obs: $T_{eff} < T_c$

→ The target qubit is refrigerated to a temperature T_{eff} colder than the available cold bath!

Experimental realization (quantum circuit)



M. A. Aamir, P. Jamet Suria, J. A. Marín Guzmán, C. Castillo-Moreno, J. M. Epstein, N. Yunger Halpern, S. Gasparinetti: Thermally driven quantum refrigerator autonomously resets a superconducting qubit. Nat. Phys. 21, 318 (2025)

Performances:

- Excited-state population reaches $3 \times 10^{-4} \pm 2 \times 10^{-4}$ (effective temperatures 22 (+2, −3) mK)
- State-of-the-art reset protocols achieve populations ranging from 8×10^{-4} to 2×10^{-3} (effective temperatures ranging from 40 mK to 49 mK)
- The relaxation time can be decreased by a factor 60 (compared to natural relaxation timescale), reaching $250ns$
- Quantum thermal machines can be useful and can also be integrated with quantum information-processing units

→ **Autonomous refrigeration:** no need of external control (remembering that external control brings in a lot of heating, see [M. Fellous-Asiani et al.: PRX Quantum 4, 040319 (2023)])

VI. Opening: work cost of quantum measurements

VI. Opening: work cost of quantum measurements

In this section, we present a research application of the concepts of heat and work for quantum systems (see Lecture 1) combined with the concept of entropy production or second law of thermodynamics (see beginning of lecture 2), for quantum systems.

- Quantum measurements (level 1): projective measurements

→ $\{P_r\}_r$ with $\begin{cases} \sum_r P_r = \mathbb{I} \\ P_r \text{ is a projector, } P_r^2 = P_r \end{cases}$

r: outcome of the measurement:

→ measurement: $\rho \xrightarrow{\text{outcome r}} \rho_{|_r} = \frac{1}{p_r} P_r \rho P_r$

probability of outcome r: $p_r = \mathrm{Tr}[\rho P_r]$

→ Observation: if P_r is of rank 1, meaning $P_r = |r\rangle\langle r|$, $\rho_{|r}$ is always a pure state

- Formalism for general quantum measurements (POVM) (level 2):

→ $\{M_r\}_r$ with $\begin{cases} \sum_r M_r^\dagger M_r = 1 \\ M_r \text{ is positive} \end{cases}$

r: outcome of the measurement:

→ measurement: $\rho \xrightarrow{\text{outcome r}} \rho_{|r} = \frac{1}{p_r} M_r \rho M_r^\dagger$

probability of outcome r: $p_r = \mathrm{Tr}[\rho M_r^\dagger M_r]$

→ Observation: if ρ is a pure state, $\rho_{|r}$ is always a pure state

proof: $\rho = |\psi\rangle\langle\psi| \implies \rho_{|r} = \frac{1}{p_r} M_r |\psi\rangle\langle\psi| M_r^\dagger \implies$

$$\begin{aligned}\mathrm{Tr}[\rho_{|r}^2] &= \frac{1}{p_r^2}\mathrm{Tr}[M_r|\psi\rangle\langle\psi|M_r^\dagger M_r|\psi\rangle\langle\psi|M_r^\dagger] \\ &= \frac{1}{p_r^2}\langle\psi|M_r^\dagger M_r|\psi\rangle\langle\psi|M_r^\dagger M_r|\psi\rangle = 1\end{aligned}$$

- Formalism for general quantum measurements (POVM) (level 3):

measurement: $\rho \overset{\text{outcome r}}{\longrightarrow} \rho_{|r} = \mathcal{O}_r \rho$

$$= \frac{1}{p_r} \sum_k M_{r,k} \rho M_{r,k}^\dagger \quad \text{with} \quad \sum_{r,k} M_{r,k}^\dagger M_{r,k} = 1$$

probability of outcome r: $p_r = \mathrm{Tr}\left[\rho \sum_k M_{r,k}^\dagger M_{r,k}\right]$

H. M. Wiseman, G. J. Milburn: Quantum Measurement and Control (2009)

Observation: even if ρ is a pure state, $\rho_{|r}$ is NOT NECESSARY a pure state!

- Formalism for general quantum measurements (POVM) (level 3):

measurement: $\rho \xrightarrow{\text{outcome r}} \rho_{|r} = \mathcal{O}_r \rho$

$$= \frac{1}{p_r} \sum_k M_{r,k} \rho M_{r,k}^\dagger \quad \text{with} \quad \sum_{r,k} M_{r,k}^\dagger M_{r,k} = 1$$

probability of outcome r: $p_r = \mathrm{Tr}\left[\rho \sum_k M_{r,k}^\dagger M_{r,k}\right]$

Inefficient measurement:

- for a given r, the different outcomes k are not distinguished
- the detector does not resolve the k degree of freedom
- The resulting state $\rho_{|r}$ is a mixed state (not full knowledge about the outcome state)

- Important observation: in general,

$$\mathrm{Tr}[\rho_{|r}H] \neq \mathrm{Tr}[\rho H] \qquad \text{and} \qquad \mathrm{Tr}\left[\sum_r \rho_{|r}H\right] \neq \mathrm{Tr}[\rho H]$$

$\longrightarrow$ quantum measurements do **inject energy into the system**

$\longrightarrow$ applications: cooling quantum systems, quantum measurement engines

?

Where does this injected energy come from?

Does it come for free?

Is it heat? Is it work?

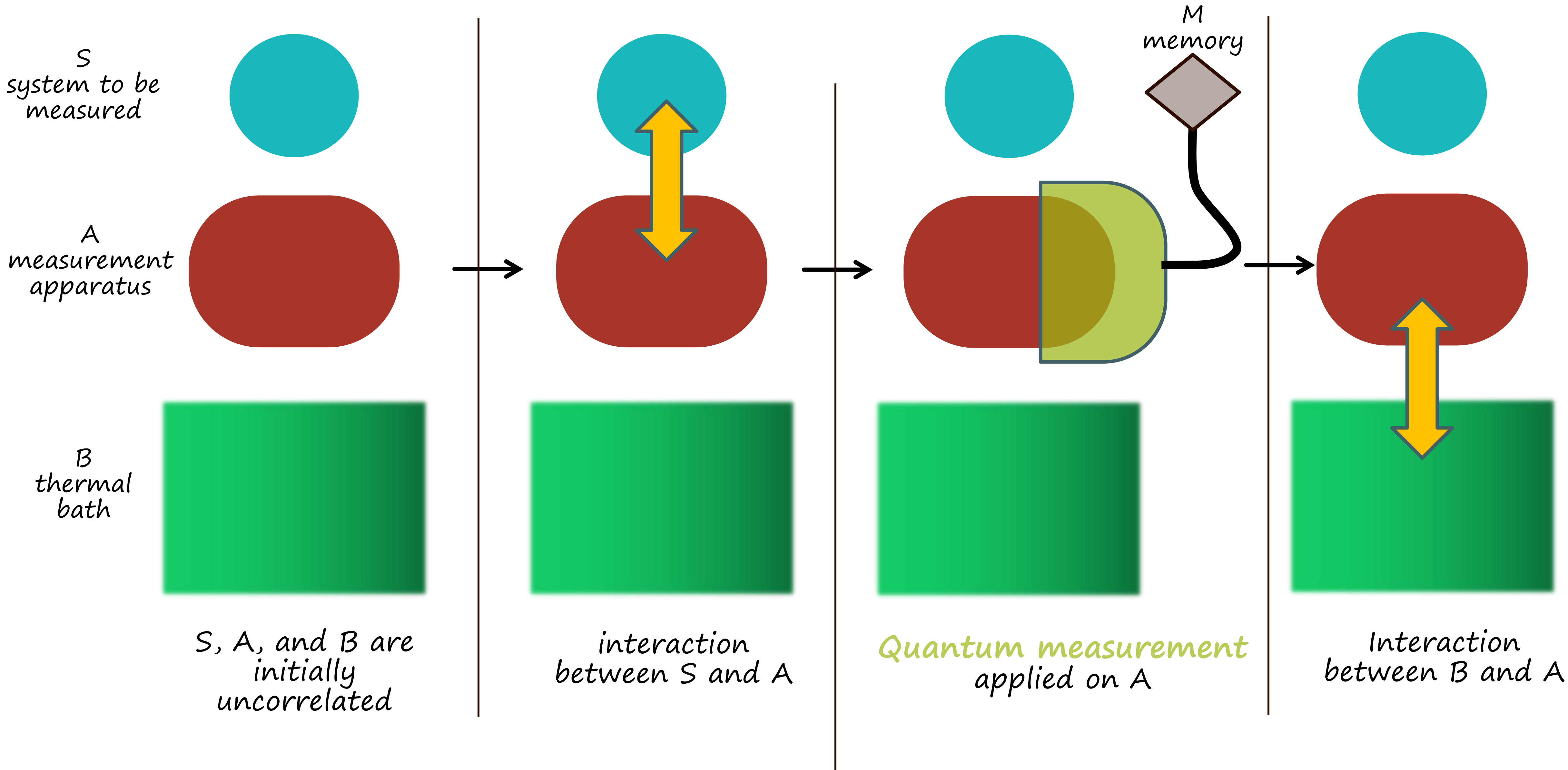
Are the envolved resources related to the quality of the measurement?

$\longrightarrow$ **Problem:**

we do not have a well established microscopic model for quantum measurement – we only have an effective description

Microscopic model – version 1

- T. Sagawa and M. Ueda, Minimal Energy Cost for Thermodynamic Information Processing: Measurement and Information Erasure, Phys. Rev. Lett. 102, 250602 (2009).
- K. Funo, Y. Watanabe, and M. Ueda, Integral quantum fluctuation theorems under measurement and feedback control, Phys. Rev. E 88, 052121 (2013)
- K. Abdelkhalek, Y. Nakata, and D. Reeb, Fundamental energy cost for quantum measurement, arXiv: 1609.06981 (2018)].
- L. Mancino, M. Sbroscia, E. Roccia, I. Gianani, F. Somma, P. Mataloni, M. Paternostro, and M. Barbieri, The entropic cost of quantum generalized measurements, npj Quantum Inf. 4, 1 (2018).

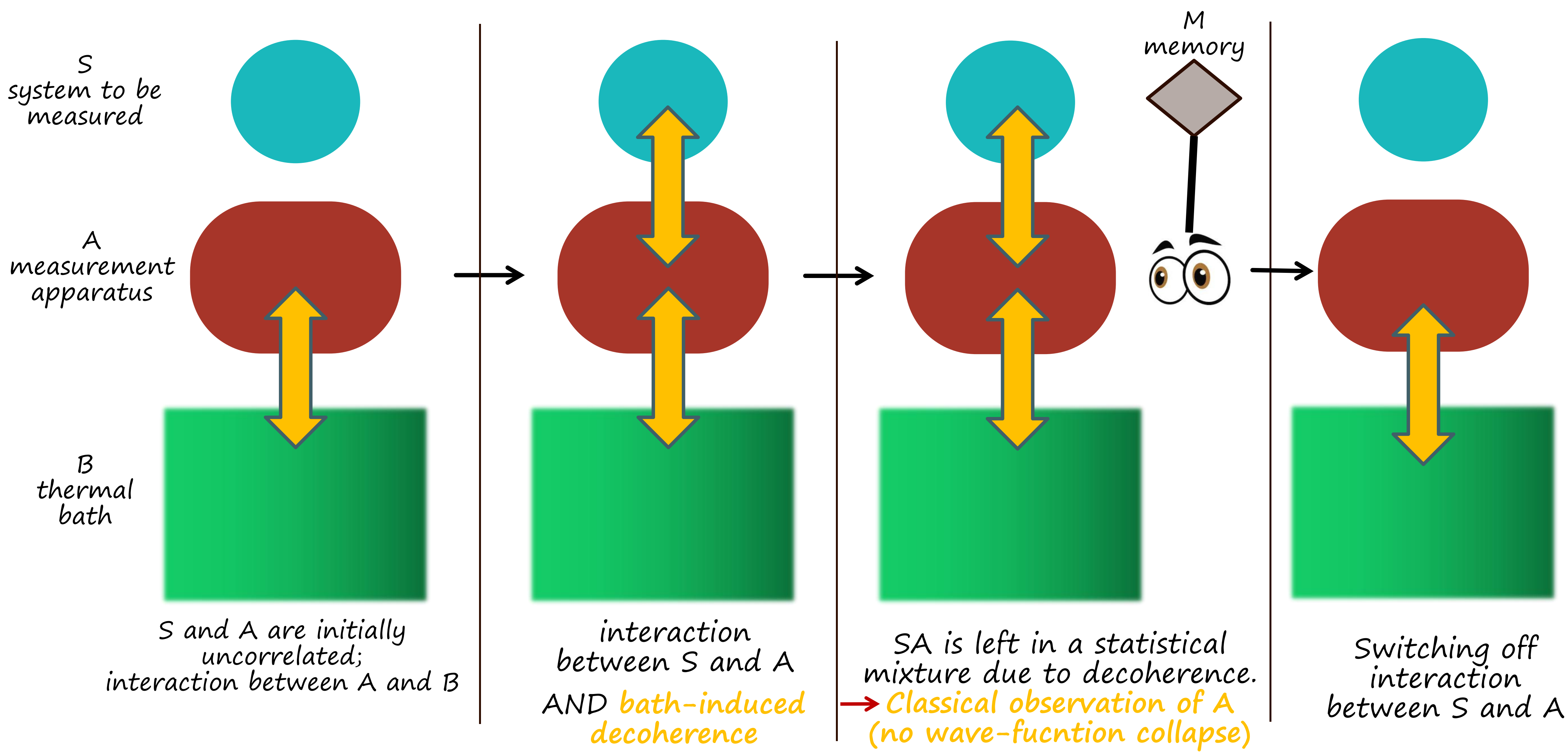


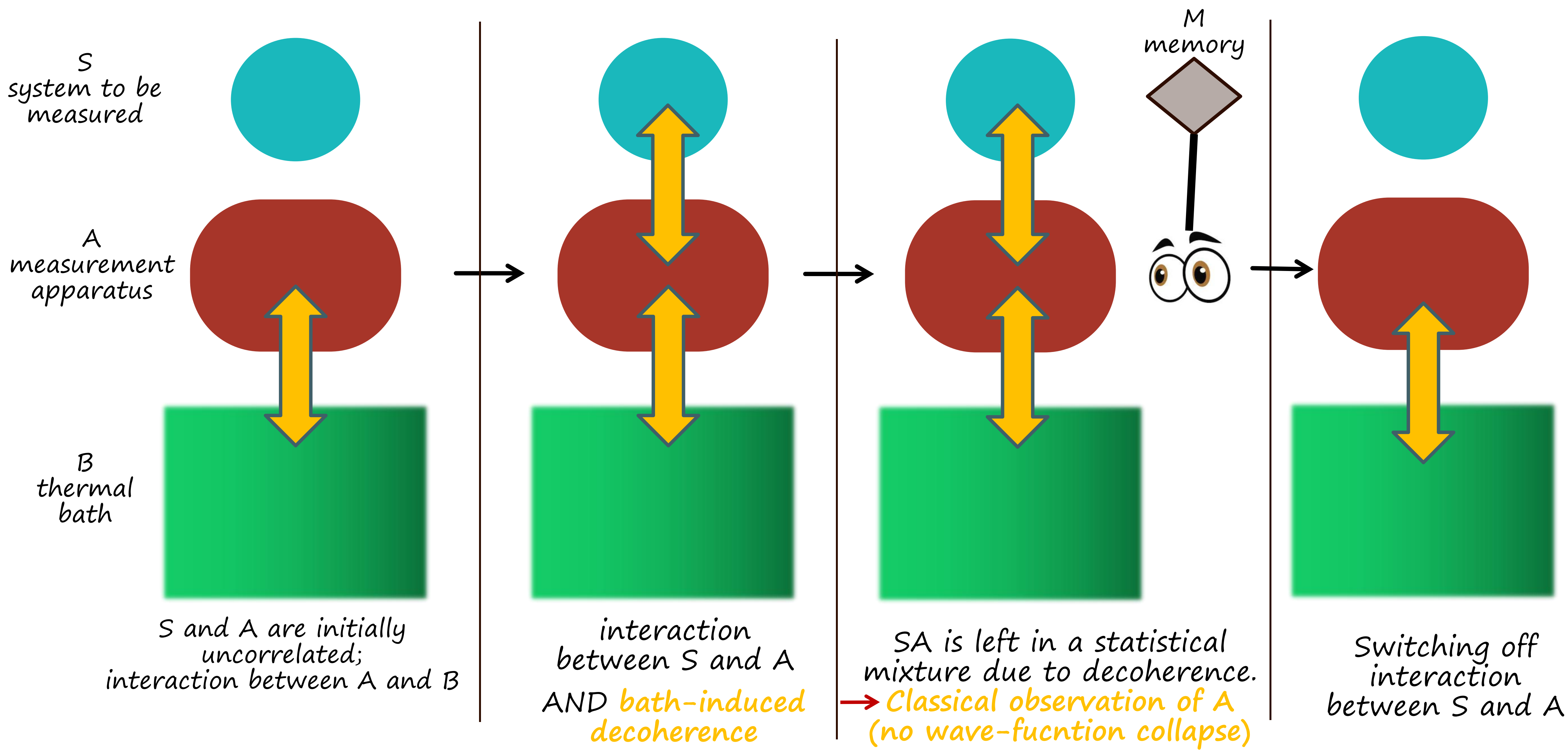
Issue: Quantum measurement applied on A

- Which resources are involved?
- What is the work involved in such measurement?

Microscopic model – version 2

- A. E. Allahverdyan, R. Balian, and T. M. Nieuwenhuizen, *Understanding quantum measurement from the solution of dynamical models*, Phys. Rep. 525, 1 (2013).
- H.-S. Goan, G. J. Milburn, H. M. Wiseman, and H. Bi Sun, *Continuous quantum measurement of two coupled quantum dots using a point contact: A quantum trajectory approach*, Phys. Rev. B 63, 125326 (2001).
- H.-P. Breuer and F. Petruccione, The Theory of Open Quantum Systems (Oxford University Press, 2007).





Main point: *no quantum measurement involved in the microscopic description*

$\Longrightarrow$

- *No hidden energetic cost*
- *Analysis is thermodynamically consistent*

Main questions:

- *What is the minimal work expenditure to realize a quantum measurement?*
- *What is the relation between the work invested and the quality of the mesurement?*
- *Do quantum measurements invovle heat exchanges?*

Minimal work cost

Work cost = work required to realize the overall operation

→ We apply the standard definition we saw in Lecture I: $W(t) = \int_0^t du \mathrm{Tr}[\rho_S(u)\dot{\hat{H}}_S(u)]$

$$\to \quad W_{\mathrm{dr}} := \int_0^{t_F} dt \mathrm{Tr}[\rho_{SAMB}(t)\dot{H}_{SAMB}(t)]$$

$$= \Delta E_{SAMB} := \mathrm{Tr}[\rho_{SAMB}(t_F)H_{SAMB}(t_F)] - \mathrm{Tr}[\rho_{SAMB}(0)H_{SAMB}(0)]$$

$V_{SA}(0) = V_{SA}(t_F) = 0$

$H_M = \mathbb{I}_M$

$$W_{\mathrm{dr}} = \Delta E_S + \Delta E_A + \Delta E_B + \underbrace{\Delta \langle V_{AB} \rangle}_{\text{variation of the coupling energy between A and B}}$$

The energy change of the bath is **lower bounded thanks to the 2d law of thermodynamics:**

→ We apply the standard definition we saw in this Lecture II: $\Sigma(t) = \Delta S_A + \beta_B \Delta E_B$

[M. Esposito, K. Lindenberg, and C. Van den Broeck, Entropy production as correlation between system and reservoir, New Journal of Physics 12, 013013 (2010)]

→ We apply that to the system S and the measurement apparatus A:

$$\Sigma(t) = \Delta S_{SA} + \beta_B \Delta E_B \geq 0 \quad \Rightarrow \quad \Delta E_B = -\frac{1}{\beta_B}\Delta S_{SA} + \Sigma(t) \geq -\frac{1}{\beta_B}\Delta S_{SA}$$

→ Substituting in $W_{\rm dr} = \Delta E_S + \Delta E_A + \Delta E_B + \Delta\langle V_{AB}\rangle$, we get

$$W_{\rm dr} \geq \Delta E_S + \Delta E_A - \frac{1}{\beta_B}\Delta S_{SA} = \Delta F_{SA}$$

Finally, using constraints imposed by the structure of the measurement, we obtain

$$W_{\mathrm{dr}} + W_{\mathrm{reset}} \geq \Delta F_S + \frac{1}{\beta_B}\xi_S + \frac{1}{\beta_B}\langle I_{S:A}\rangle$$

Work required to reset AM

$W_{\mathrm{reset}} \geq F_{AM}(0) - F_{AM}(t_F)$

$$\xi_S = S[\rho_S(t_F)] - \sum_r p_r S[\rho_{S|_r}(t_F)] \in [0; H(p_r)]$$

- Average gain of information when reading the output versus not reading it
- Final information shared with the memory, $I_{S:M}(t_F)$
- level of distinguishability of the $\rho_{S|_r}(t_F)$ (Holevo information)
- Characterizes the level of efficiency of the measurement

$\Delta F_S = \Delta E_S - \beta_B \Delta S_S$

variation of (non-equilibrium) free energy

$$\langle I_{S:A}\rangle = \sum_r p_r \Big\{ S[\rho_{S|_r}(t_F)] + S[\rho_{A|_r}(t_F)] - S[\rho_{SA|_r}(t_F)] \Big\}$$

Average residual correlations between S and A

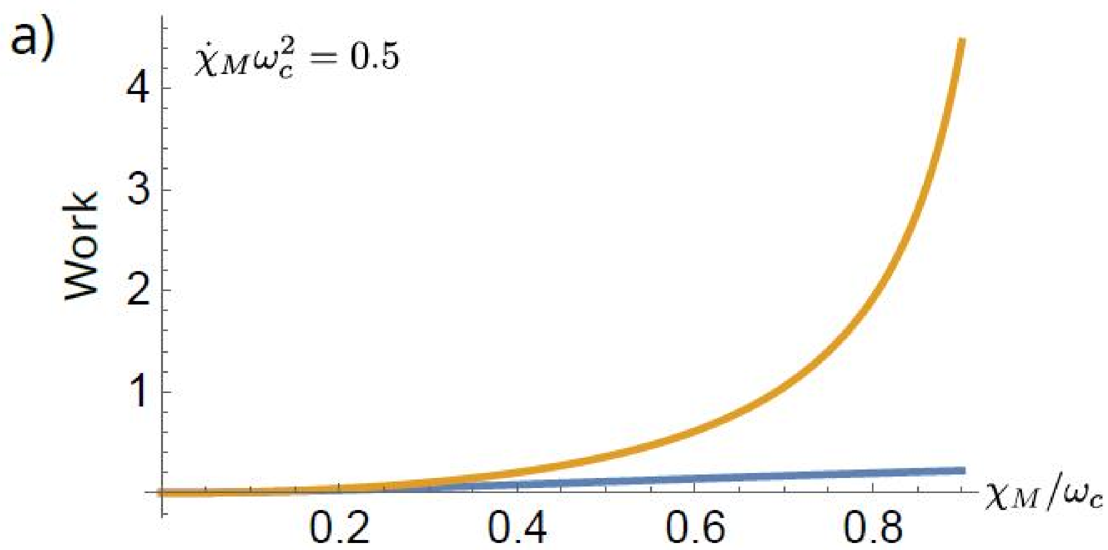
C. L. Latune, C. Elouard: A thermodynamically consistent approach to the energy costs of quantum measurements. Quantum 9, 1614 (2025)

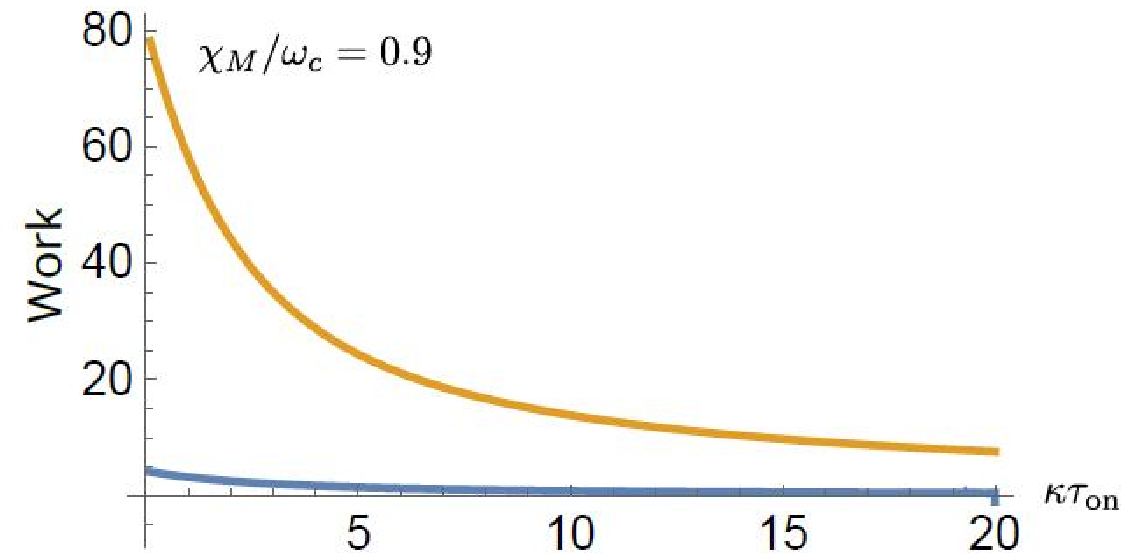
3.2 Consequences

$$W_{\mathrm{dr}} + W_{\mathrm{reset}} \geq \Delta F_S + \frac{1}{\beta_B}\xi_S + \frac{1}{\beta_B}\langle I_{S:A}\rangle$$

- Additional terms with respect to local manipulation of S ($W \geq \Delta F_S$)
- Residual correlations bring extra cost (represents energetic losses)
- Trade-off cost Vs quality (efficiency)
- Energy gained by the system during a measurement does not come for free, it costs work ⇒ this has some implications for quantum measurement engines
- Saturation of the lower bound (⇒ measurement with zero entropy production)?

Some simulations





strength of the coupling system-apparatus
(determines the efficiency of the measurement)

velocity of the measurement

Double trade-off:

<u>measurement efficiency</u> Vs <u>work cost</u> Vs <u>velocity</u>

C. L. Latune, C. Elouard: A thermodynamically consistent approach to the energy costs of quantum measurements. Quantum 9, 1614 (2025)