South African Quantum Technology Initiative

NITheCS
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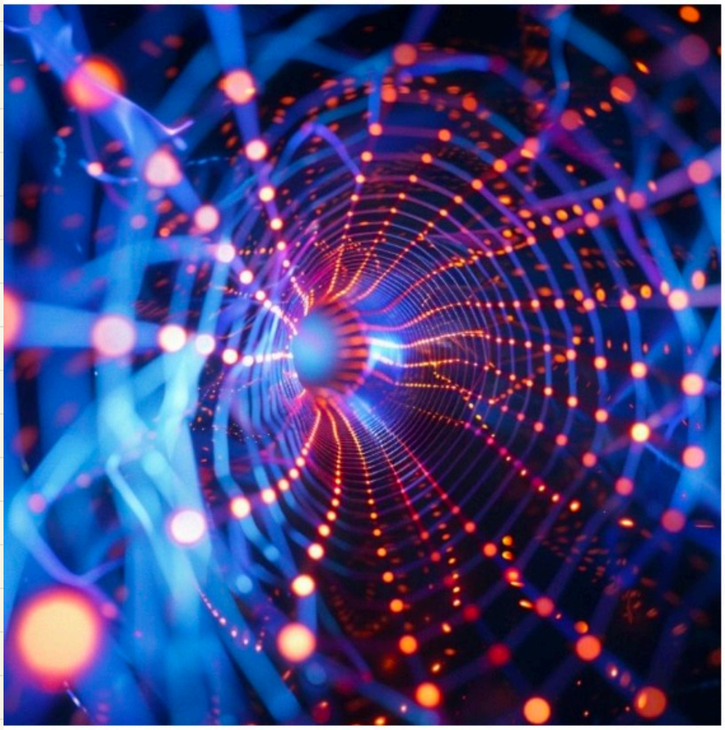
-- 33rd Chris Engelbrecht Summer School 2025 --



Theoretical Foundations of Quantum Science and Quantum Technologies

STIAS, Stellenbosch, South Africa

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Lectures on

Open quantum systems

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Calculation of $c_1(t) \Leftrightarrow \gamma(t), S(t)$

<u>Recall</u>: $\phi(t) = c_0\psi_0 + c_1(t)\psi_1 + \sum_k d_k(t)\chi_k$

$$\psi_0 = |0\rangle_S \otimes |vac\rangle_B$$
$$\psi_1 = |1\rangle_S \otimes |vac\rangle_B$$
$$\chi_k = |0\rangle_S \otimes |k\rangle_B$$

$$i\partial_t \phi(t) = H_I(t)\,\phi(t)$$

$\sigma_+(t)B(t) + \sigma_-(t)B(t)^*$

$\sigma_+(t) = e^{i\omega_0 t}\sigma_+$

$B(t) = \sum_k g_k e^{-i\omega_k t} b_k$

$\langle\psi_1|$: $\quad i\dot{c}_1(t) = \langle\psi_1, H_I(t)\phi(t)\rangle$

$$= \langle\psi_1, \sigma_+(t)B(t)\phi(t)\rangle$$

$$= \sum_k g_k e^{i(\omega_0-\omega_k)t}\, d_k(t)$$

$\langle\chi_k|$: $\quad i\dot{d}_k(t) = \bar{g}_k e^{-i(\omega_0-\omega_k)t} c_1(t)$

Take $d_k(0)=0$: $\quad d_k(t) = -i\bar{g}_k \int_0^t e^{-i(\omega_0-\omega_k)s} c_1(s)\,ds$

$$\Rightarrow \quad \dot{c}_1(t) = -\int_0^t f(t-s)\,c_1(s)\,ds$$

easy

where $\quad f(\tau) = \sum_k |g_k|^2 e^{-i\omega_k\tau} = \mathrm{tr}\big(|vac\rangle\langle vac|\; B(\tau)B^*\big)$

vacuum correlation function

Continuous mode limit

$$\sum_k |g_k|^2 e^{-i\omega_k \tau}$$

$$= \sum_\omega e^{-i\omega\tau} \underbrace{\sum_{\{k : \omega_k = \omega\}} |g_k|^2}_{n(\omega)\ |g(\omega)|^2}$$

$$= \sum_\omega \frac{n(\omega)}{\Delta\omega} e^{-i\omega\tau} |g(\omega)|^2 \ \Delta\omega$$

$$\longrightarrow \int_0^\infty \underline{\rho(\omega)}\, |g(\omega)|^2 e^{-i\omega\tau} d\omega = C(\tau)$$

density of modes

correl. function

- $C(\tau)$ decays in τ, speed depends on regularity of $\rho(\omega)\,|g(\omega)|^2$

- $C(\tau)$ is Fourier transform of the <u>spectral density</u>

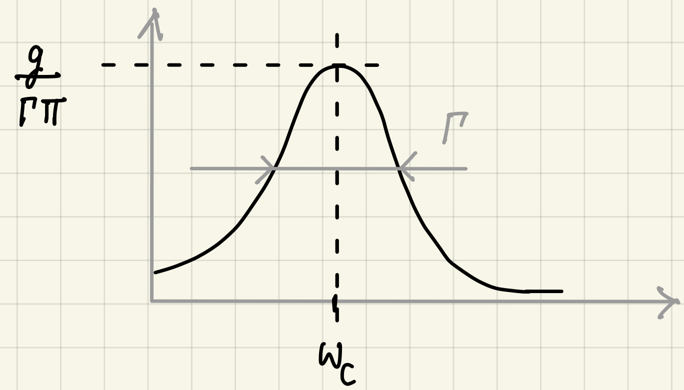
$$J(\omega) = \sqrt{2\pi}\, \rho(\omega)\, |g(\omega)|^2$$

Physics :

$$J(\omega) = \frac{g}{\pi} \frac{\Gamma}{(\omega-\omega_c)^2+\Gamma^2}$$ Lorentzian $(\Gamma>0)$

width

[Garraway, PRA 96, 97]



Amplitude of $\psi_1 = |1\rangle_S \otimes |vac\rangle_B$:

$$c_1(t) = c_1(0)\, e^{-(\Gamma - i\Delta)t/2} \left[\cosh\left(\frac{Rt}{2}\right) + \frac{\Gamma - i\Delta}{R} \sinh\left(\frac{Rt}{2}\right) \right]$$

atom

$$\Delta = \omega_0 - \omega_c$$ cavity (detuning)

$$R = \sqrt{(\Gamma - i\Delta)^2 - 4g/\sqrt{2\pi}}$$ (principal branch)

[Li-Zou-Shao, PRA 81, 062124 (2020)]

Recall:

- Initially cavity in vacuum state
- Atom density matrix

$$\rho_S(t) = \begin{pmatrix} |c_1(t)|^2 & e^{-i\omega_0 t}\,\bar{c}_0\, c_1(t) \\ e^{i\omega_0 t}\, c_0\, \overline{c_1(t)} & 1 - |c_1(t)|^2 \end{pmatrix}$$

- Lorentzian spectral density $\rightarrow$ explicit $c_1(t)$

Resonant case $\Delta = \omega_0 - \omega_c = 0$:

$$c_1(t) = c_1(0)\, e^{-\Gamma t/2}\left[\cosh\left(\frac{R_0 t}{2}\right) + \frac{\Gamma}{R_0}\sinh\left(\frac{R_0 t}{2}\right)\right]$$

$$R_0 = \sqrt{\Gamma^2 - 4g/\sqrt{2\pi}}$$

parameters determining the spectral density of noise.

$\Gamma > 4g/\sqrt{2\pi}$: $R_0 > 0$

Then

$$\frac{\dot{c}_1(t)}{c_1(t)} = -\frac{1}{2}\gamma(t) - \cancelto{0}{\frac{i}{2}\delta(t)} \quad \in \mathbb{R}$$

and

$$c_1(t) = c_0(t)\cdot e^{-\frac{1}{2}\int_0^t \gamma(\tau)\,d\tau}$$

$$\gamma(t) = \frac{4g}{\sqrt{2\pi}} \frac{\sinh(R_0 t/2)}{R_0 \cosh\left(\frac{R_0 t}{2}\right) + \Gamma \sinh\left(\frac{R_0 t}{2}\right)}$$

① If $R_0 t/2 \ll 1$:

$$\gamma(t) \approx \frac{4g}{\sqrt{2\pi}} \frac{\cancel{R_0} t/2}{\cancelto{1}{R_0} + \Gamma \frac{\cancel{R_0} t}{2}}$$

$$\approx \sqrt{\frac{2}{\pi}}\, g t \qquad (\Gamma t/2 \ll 1)$$

② If $R_0 t/2 \gg 1$:

$$\gamma(t) \approx \frac{4g}{\sqrt{2\pi}} \frac{1}{R_0 + \Gamma} \overset{g \ll \Gamma^2}{\approx} \frac{2g}{\sqrt{2\pi}} \frac{1}{\Gamma}$$

independent of time.

Summary: Dissipative Jaynes-Cummings model

- Reduced dynamics of system shows irreversibility (time decay) due to coupling with a continuum of oscillator modes.

- $$\rho_s(t) \xrightarrow{t\to\infty} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = |0\rangle\langle 0| \qquad \text{(ground state)}$$

  "speed of convergence" $\sim e^{-\frac{1}{2}\int_0^t \gamma(s)\,ds}$

  $$\begin{cases} \gamma(s) \sim \text{const.}, & t \text{ large} \\ \gamma(s) \sim \gamma_0 t, & t \text{ small} \end{cases}$$

- Equation of motion of reduced dynamics:

$$\partial_t \rho(t) = -i\left[H_{eff}(t), \rho(t)\right] + \gamma(t)\left\{\sigma_- \rho(t)\sigma_+ - \frac{1}{2}\sigma_+\sigma_-\rho(t) - \frac{1}{2}\rho(t)\sigma_+\sigma_-\right\}$$

(Resonant case: $H_{eff}(t) = H_s = \omega_0\, \sigma_+\sigma_-$)

- For large times, $\gamma(t) = \gamma$, constant.

$$\partial_t \rho(t) = -i\,[H_S, \rho(t)] + \gamma\, \mathcal{D}[\rho(t)]$$

"dissipator"

$$= \mathcal{L}[\rho(t)]$$

$\mathcal{L}$: Lindblad operator

$$\Rightarrow \qquad \rho(t) = e^{t\mathcal{L}} \rho(0)$$

evolution (semi) group

Two questions

$\boxed{1}$ General form of superoperator $\mathcal{L}$ s.t. $\rho(t) = e^{t\mathcal{L}}\rho(0)$ is evolution of a reduced density matrix?

$\boxed{2}$ Under what conditions is $e^{t\mathcal{L}}$ a good approximation to the true dynamics of the reduced dmat? MARKOVIAN APPROX.

Part 2

CPTP maps

CPTP maps

(completely positive trace preserving)

Physical motivation:

$\mathcal{H}, \mathcal{H}_R$: Hilbert spaces

$\Omega \in \mathcal{H}_R$: normalized vector (pure state)

$\mathcal{U}$: unitary on $\mathcal{H} \otimes \mathcal{H}_R$ (quantum channel)

Define map Φ on bounded operators $X \in \mathcal{B}(\mathcal{H}_S)$:

$$\Phi(X) = \mathrm{tr}_R \left[\mathcal{U} \left(X \otimes |\Omega\rangle\langle\Omega| \right) \mathcal{U}^* \right]$$

partial trace

think of $\mathcal{U}$ as e^{itH} (t fixed)

Partial trace: $A = B \otimes C$ operator on $\mathcal{H} \otimes \mathcal{H}_R$

$$\longrightarrow \mathrm{tr}_R A = B \cdot \underbrace{\mathrm{tr}_R C}_{\in \mathbb{C}}$$

Extend $\mathrm{tr}_R : \mathcal{B}(\mathcal{H} \otimes \mathcal{H}_R) \longrightarrow \mathcal{B}(\mathcal{H})$ by linearity.

$$U = \sum_{ij} u_{ij}\, P_i \otimes Q_j \qquad (u_{ij} \in \mathbb{C},\ P_i, Q_j \text{ operators})$$

$$\rightarrow \Phi(X) = \mathrm{tr}_R\, U \left(X \otimes |\Omega\rangle\langle\Omega| \right) U^*$$

$$= \sum_{ij} \sum_{kl} u_{ij}\, \bar{u}_{kl}\, \mathrm{tr}_R \left(P_i \otimes Q_j \right) \left(X \otimes |\Omega\rangle\langle\Omega| \right) P_k^* \otimes Q_l^*$$

$$= \sum_{ijkl} u_{ij}\, \bar{u}_{kl}\, P_i X P_k^*\, \underbrace{\mathrm{tr}\left(Q_j |\Omega\rangle\langle\Omega| Q_l^* \right)}_{\sum_\alpha \langle f_\alpha,\, Q_j |\Omega\rangle\langle\Omega| Q_l^* f_\alpha \rangle}$$

where $\{f_\alpha\}$ is any ONB of $\mathcal{H}_R$

$$= \sum_\alpha \left(\sum_{ij} u_{ij} \langle f_\alpha, Q_j \Omega \rangle P_i \right) X \left(\sum_{kl} \bar{u}_{kl} \langle \Omega, Q_l^* f_\alpha \rangle P_k^* \right)$$

Define Kraus operators $K_\alpha = \sum_{ij} u_{ij} \langle f_\alpha, Q_j \Omega \rangle P_i$

$$\rightarrow \boxed{\Phi(X) = \sum_\alpha K_\alpha X K_\alpha^*}$$