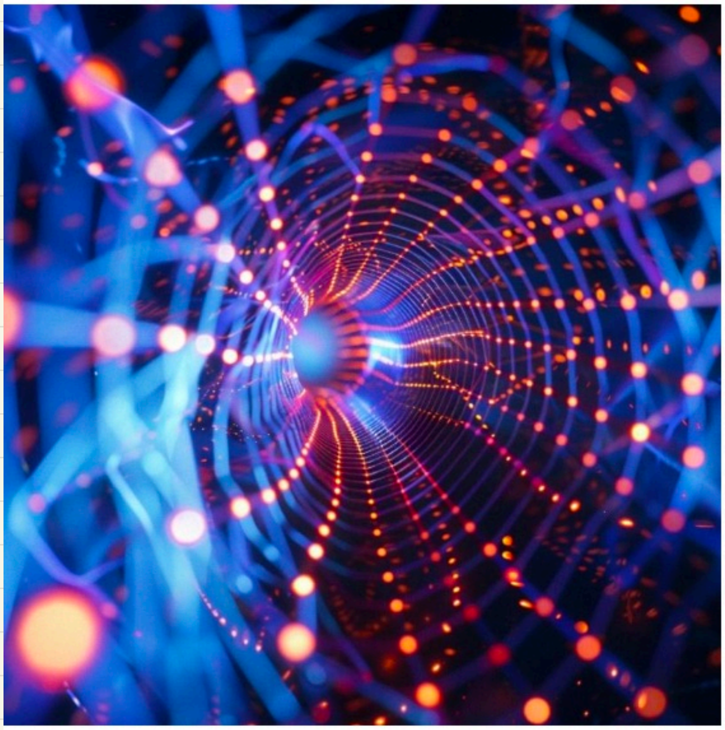
SA QuTI South African Quantum Technology Initiative

NITheCS National Institute for Theoretical and Computational Sciences

International Year of Quantum Science and Technology

-- 33rd Chris Engelbrecht Summer School 2025 --



Theoretical Foundations of Quantum Science and Quantum Technologies

STIAS,
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Lectures on

Open quantum systems

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Part 1

Open quantum systems by example:

The open Jaynes-Cummings model

① Formalism of quantum theory

- $\underline{\text{state}}$: normalized vector ψ of a Hilbert space $\mathcal{H}$

E.g.

$\mathcal{H}$:

- $\mathbb{C}^2$ — qubit (spin)
- $L^2(\mathbb{R}^3, d^3x)$ — particle in $\mathbb{R}^3$
- $\mathcal{F} = \mathbb{C} \bigoplus_{n=1}^{\infty} \mathfrak{h}^{\otimes n}$ — quantum field (Fock space)

- $\underline{\text{Composite system}}$ Hilbert space : product of individual ones

E.g.

$\mathcal{H}$:

- $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$ — qubit register
- $\mathbb{C}^2 \otimes \mathcal{F}$ — atom - radiation complex

- **Observables** are self-adjoint operators A on $\mathcal{H}$. A measurement readout is always an eigenvalue of A. The readout for fixed A and ψ is **random** having average

$$\langle A \rangle_\psi = \langle \psi, A\psi \rangle$$

$\boxed{\text{E.g.}}$

A:

- $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = |1\rangle\langle 1|$ — qubit value
- $x, \quad -i\hbar\nabla_x$ — position, momentum of a particle

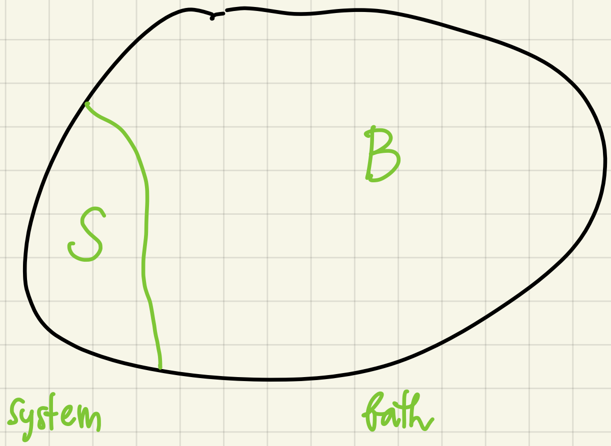
- **Dynamics** : given by the **Schrödinger equation**

$$i\hbar \frac{d}{dt}\psi = H\psi$$

H: **Hamiltonian** (energy observable)

$$\Leftrightarrow \quad \psi_t = e^{-itH}\psi_0$$

Reduced state



$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B$

$\psi \in \mathcal{H}$: generally entangled

Measure observables $A \in \mathcal{B}(\mathcal{H}_S)$ of S alone:

$$\langle \psi, (A \otimes \mathbb{1}_B)\psi \rangle_{\mathcal{H}} = \mathrm{tr}_{\mathcal{H}_S}(\rho A)$$

defines the <u>density matrix</u>, or <u>reduced state</u>

$$\rho = \rho^* \geq 0, \qquad \mathrm{tr}_{\mathcal{H}_S}\, \rho = 1, \qquad \rho = \mathrm{tr}_{\mathcal{H}_B} |\psi\rangle\langle\psi|$$

partial trace

$$\langle \psi, (A \otimes \mathbb{1})\psi \rangle = \mathrm{tr}_{\mathcal{H}_S \otimes \mathcal{H}_B} \Big(|\psi\rangle\langle\psi| \, (A \otimes \mathbb{1}) \Big)$$

$$= \mathrm{tr}_{\mathcal{H}_S} \Big[\big(\mathrm{tr}_{\mathcal{H}_B} |\psi\rangle\langle\psi|\big) \, A \Big]$$

Revisit our notion of state: $\rho = \rho^* \geq 0$ on $\mathcal{H}$, $\operatorname{tr} \rho = 1$.

Expectation of observable $\langle A \rangle = \operatorname{tr} \rho A$.

($\rho = |\psi\rangle\langle\psi|$ special case.)

<u>What is the evolution of the reduced state?</u>

$$\rho_t = \operatorname{tr}_B \left(|\psi_t\rangle\langle\psi_t| \right) = \operatorname{tr}_B \left(e^{-itH} |\psi\rangle\langle\psi| e^{itH} \right)$$

- NOT of the form $e^{-itH_{eff}} \rho_0 e^{itH_{eff}}$, not a group
- But also not completely arbitrary

  $\hookrightarrow$ reduction of Schrödinger dynamics gives rise to special structure (CPTP)

The dissipative Jaynes-Cummings model

atom in a cavity

$\nearrow$ $\nwarrow$

2-level system

$\{ |1\rangle, |0\rangle \}$

$\sigma_+ = |1\rangle\langle 0|$

$\sigma_- = |0\rangle\langle 1|$

harmonic oscillator

$b_k^+ , \; b_k \quad , \quad k = 1, 2, \ldots$

$[b_k^*, b_l] = -\delta_{kl}$

" bath "

$$H = \underbrace{H_S + H_B}_{H_0} + H_I$$

$$H_S = \omega_0 \, |1\rangle\langle 1| = \omega_0 \, \sigma_+ \sigma_-$$

$$H_B = \sum_k \omega_k \, b_k^* b_k$$

$$H_I = \sigma_+ \otimes B + \sigma_- \otimes B^*$$

$\rightarrow \sum_k g_k b_k \, , \quad g_k \in \mathbb{C} .$

Dynamics of full complex (atom plus bath)

$$i\partial_t \psi(t) = H\psi(t)$$

Interaction picture: $\phi(t) := e^{itH_0}\psi(t)$

$$\Rightarrow \qquad i\partial_t \phi(t) = H_I(t)\,\phi(t)$$

$$H_I(t) = e^{itH_0} H_I e^{-itH_0}$$

$$= \sigma_+(t)\,B(t) + h.c.$$

$e^{i\omega_0 t}\sigma_+$ $\qquad\qquad$ $\sum_k g_k e^{-i\omega_k t} b_k$.

The number of excitations is conserved

$$[H, N] = 0$$

where total number operator is $N = |1\rangle\langle 1| + \sum_k b_k^* b_k$

$\Rightarrow$ spectral subspaces of N are invariant under dynamics

Set $|vac\rangle_B = |000\cdots0\rangle_B$ $\quad$ ($b_k|0\rangle = 0$ G.S.)

$|k\rangle_B = b_k^* |vac\rangle_B$ $\quad$ (1 excitation, osci k)

and

$$\psi_0 = |0\rangle_S \otimes |vac\rangle_B$$

$$\psi_1 = |1\rangle_S \otimes |vac\rangle_B$$

$$\chi_k = |0\rangle_S \otimes |k\rangle_B$$

Take initial state with at most $N=1$ excitations :

$$\phi(0) = c_0 \psi_0 + c_1(0)\psi_1 + \sum_k d_k(0)\,\chi_k$$

Then $\phi(t)$ also has at most $N=1$ excitations (all t)

$$\phi(t) = c_0 \psi_0 + c_1(t)\psi_1 + \sum_k d_k(t)\,\chi_k$$

The reduced atom density matrix (interaction picture) is,

$$\rho_{S,I}(t) = tr_B\left(|\phi(t)\rangle\langle\phi(t)|\right)$$

$$= \begin{pmatrix} |c_1(t)|^2 & \bar{c}_0\, c_1(t) \\ c_0\, \overline{c_1(t)} & 1-|c_1(t)|^2 \end{pmatrix}$$ (Exercise 1)

Written as matrix in basis $\{|1\rangle, |0\rangle\}$ of atom.

We get an equation of evolution for $\rho_{S,I}$:

$$\partial_t \rho_{S,I}(t) = \begin{pmatrix} \partial_t |c_1(t)|^2 & \bar{c}_0\, \dot{c}_1(t) \\ c_0\, \overline{\dot{c}_1(t)} & -\partial_t |c_1(t)|^2 \end{pmatrix}$$

Suppose $c_1(0) \neq 0$. $\dfrac{\dot{c}_1(t)}{c_1(t)}$ is complex number, so

$$\frac{\dot{c}_1(t)}{c_1(t)} = -\frac{1}{2}\gamma(t) - i\,\frac{1}{2}S(t)$$ (defines γ, S)

$$\rightarrow \quad c_1(t) = e^{-\frac{1}{2}\int_0^t (\gamma(\tau) + iS(\tau))d\tau}\, c_1(0)$$

Then $\quad \partial_t |c_1(t)|^2 = -\gamma(t)\, |c_1(t)|^2 \quad$ and

$$\partial_t \rho_{S,I}(t) = \begin{pmatrix} -\gamma(t)\, |c_1(t)|^2 & -\frac{1}{2}\bar{c}_0 \left(\gamma(t) + i S(t)\right) c_1(t) \\ \frac{1}{2} c_0 \left(\gamma(t) - i S(t)\right) \overline{c_1(t)} & \gamma(t)\, |c_1(t)|^2 \end{pmatrix}$$

Excercise 2

$$= -\frac{i}{2} S(t) \left[\sigma_+ \sigma_-, \rho_{S,I}(t)\right] + \gamma(t) \left\{ \sigma_- \rho_{S,I}(t) \sigma_+ - \frac{1}{2} \sigma_+ \sigma_- \rho_{S,I}(t) - \frac{1}{2} \rho_{S,I}(t) \sigma_+ \sigma_- \right\}$$

Undo the interaction picture:

$$\rho_S(t) = e^{-itH_S} \rho_{S,I}\, e^{itH_S}$$

$$\Rightarrow \boxed{\partial_t \rho_S(t) = -i\left[H_S + \frac{1}{2} S(t) \sigma_+ \sigma_-, \rho_S(t)\right] - \gamma(t) \left\{ \sigma_- \rho_S(t) \sigma_+ - \frac{1}{2} \sigma_+ \sigma_- \rho_S(t) - \frac{1}{2} \rho_S(t) \sigma_+ \sigma_- \right\}}$$