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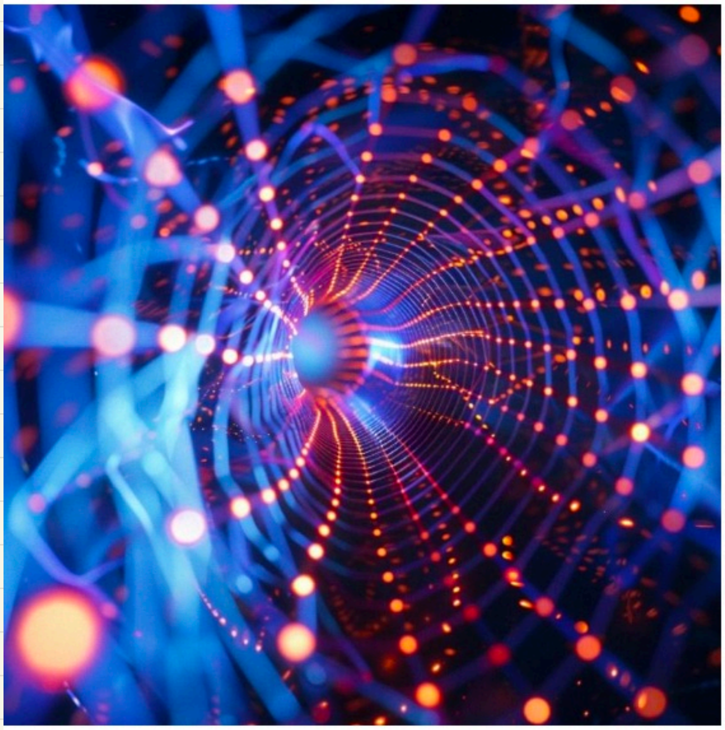

Theoretical Foundations of Quantum Science and Quantum Technologies

STIAS,
Stellenbosch, South Africa

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Lectures on

Open quantum systems

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Part 2

CPTP maps

CPTP maps

(completely positive trace preserving)

Physical motivation:

$\mathcal{H}$, $\mathcal{H}_R$: Hilbert spaces

$\Omega \in \mathcal{H}_R$: normalized vector (pure state)

$\mathcal{U}$: unitary on $\mathcal{H} \otimes \mathcal{H}_R$ (quantum channel)

Define map Φ on bounded operators $X \in \mathcal{B}(\mathcal{H}_S)$:

$$\Phi(X) = \operatorname{tr}_R \left[\mathcal{U} \left(X \otimes |\Omega\rangle\langle\Omega| \right) \mathcal{U}^* \right]$$

↖ partial trace

↑ think of $\mathcal{U}$ as e^{itH} (t fixed)

Partial trace: $A = B \otimes C$ operator on $\mathcal{H} \otimes \mathcal{H}_R$

$\longrightarrow \operatorname{tr}_R A = B \cdot \underbrace{\operatorname{tr}_R C}_{\in \mathbb{C}}$

Extend $\operatorname{tr}_R : \mathcal{B}(\mathcal{H} \otimes \mathcal{H}_R) \longrightarrow \mathcal{B}(\mathcal{H})$ by linearity.

$$\mathcal{U} = \sum_{ij} u_{ij} \, P_i \otimes Q_j \qquad (u_{ij} \in \mathbb{C}, \; P_i, Q_j \text{ operators})$$

$$\to \Phi(X) = \operatorname{tr}_R \, \mathcal{U} \left(X \otimes |\Omega\rangle\langle\Omega| \right) \mathcal{U}^*$$

$$= \sum_{ij} \sum_{kl} u_{ij} \, \bar{u}_{kl} \, \operatorname{tr}_R \left(P_i \otimes Q_j \right) \left(X \otimes |\Omega\rangle\langle\Omega| \right) P_k^* \otimes Q_l^*$$

$$= \sum_{ijkl} u_{ij} \, \bar{u}_{kl} \, P_i X P_k^* \, \underbrace{\operatorname{tr}\left(Q_j |\Omega\rangle\langle\Omega| Q_l^* \right)}$$

$$\sum_\alpha \langle f_\alpha, \, Q_j |\Omega\rangle\langle\Omega| Q_l^* f_\alpha \rangle$$

where $\{f_\alpha\}$ is any ONB of $\mathcal{H}_R$

$$= \sum_\alpha \left(\sum_{ij} u_{ij} \langle f_\alpha, Q_j \Omega \rangle P_i \right) X \left(\sum_{kl} \bar{u}_{kl} \langle \Omega, Q_l^* f_\alpha \rangle P_k^* \right)$$

Define Kraus operators $K_\alpha = \sum_{ij} u_{ij} \langle f_\alpha, Q_j \Omega \rangle P_i$

$$\to \quad \boxed{\Phi(X) = \sum_\alpha K_\alpha X K_\alpha^*}$$

What are the properties of K_α ?

(1) $\operatorname{tr} \Phi(X) = \operatorname{tr}_{\mathcal{H}\otimes\mathcal{H}_R} \cancel{\mathcal{U}} \left(X \otimes |\Omega\rangle\langle\Omega|\right) \cancel{\mathcal{U}^*}$

$= \operatorname{tr} X$ — Φ is trace-preserving

(2) $\operatorname{tr} \Phi(X) = \operatorname{tr}\left(\sum_\alpha K_\alpha X K_\alpha^*\right)$

$= \operatorname{tr}\left(\sum_\alpha K_\alpha^* K_\alpha X\right)$

(1) & (2) $\Rightarrow$ $\operatorname{tr}\left(\sum_\alpha K_\alpha^* K_\alpha - \mathbb{1}\right) X = 0, \quad \forall X \in \mathcal{B}(\mathcal{H}_S)$

This implies (exercise)

$$\sum_\alpha K_\alpha^* K_\alpha = \mathbb{1}.$$

Consider $\Phi \otimes \mathbb{1}_n$ acting on $\mathcal{H}_S \otimes \mathbb{C}^n$, $n \geq 1$.

Then $\Phi \otimes \mathbb{1}_n = \sum_\alpha K_\alpha \otimes \mathbb{1}_n \, (\cdot) \, K_\alpha^* \otimes \mathbb{1}_n$

If $A \in \mathcal{B}(\mathcal{H}_S \otimes \mathbb{C}^n)$ is a positive operator, $A \geq 0$, then

$$\Phi \otimes \mathbb{1}_n (A) = \sum_\alpha (K_\alpha \otimes \mathbb{1}_n) \, A \, (K_\alpha^* \otimes \mathbb{1}_n)$$

is also a positive operator on $\mathcal{H}_S \otimes \mathbb{C}^n$ (prove it!)

$\Rightarrow$ $\Phi \otimes \mathbb{1}_n$ is positivity preserving $\forall n \geq 1$.

A map Φ s.t. $\Phi \otimes \mathbb{1}_n$ positivity preserving $\forall n \geq 1$ is called completely positive (CP).

All in all, Φ is completely positive and trace preserving "Φ is CPTP".

Amazing fact: Every CPTP map is of Kraus form!

We now show the amazing fact!

Let Φ be CPTP on $\mathcal{B}(\mathcal{H})$.

We take $\psi \in \mathcal{H}$ and investigate $\Phi(|\psi\rangle\langle\psi|)$.

Let $\{e_i\}$ an ONB of $\mathcal{H}$ and let $\mathcal{C}$ be the anti-linear map taking complex conjugation of components in the basis $\{e_i\}$.

Then $\langle \mathcal{C}\psi, \mathcal{C}\varphi \rangle = \langle \varphi, \psi \rangle$ $\forall \psi, \varphi \in \mathcal{H}$ and $\mathcal{C} e_i = e_i$.

With $\psi = \sum_j \langle e_j, \psi \rangle e_j$:

$$\Phi(|\psi\rangle\langle\psi|) = \sum_{jk} \underbrace{\langle e_j, \psi \rangle}_{\langle \mathcal{C}\psi, e_j \rangle} \Phi(|e_j\rangle\langle e_k|) \underbrace{\overline{\langle e_k, \psi \rangle}}_{\langle e_k, \mathcal{C}\psi \rangle}$$

$$= \sum_{jk} \langle \mathcal{C}\psi, e_j \rangle \langle e_k, \mathcal{C}\psi \rangle \Phi(|e_j\rangle\langle e_k|)$$

$$= \sum_{jk} \operatorname{tr}_2 \left(\Phi \otimes |\mathcal{C}\psi\rangle\langle\mathcal{C}\psi|\right) \left(\underbrace{|e_j\rangle\langle e_k| \otimes |e_j\rangle\langle e_k|}\right)$$

$|e_j \otimes e_j\rangle\langle e_k \otimes e_k|$

$$= tr_2 \left(\Phi \otimes |e\psi\rangle\langle e\psi|\right)\left(|v\rangle\langle v|\right)$$

where $v = \sum_j e_j \otimes e_j$. Since Φ is CP:

$$(\Phi \otimes 1)(|v\rangle\langle v|) = \sum_\alpha |s_\alpha\rangle\langle s_\alpha|, \quad \text{some } s_\alpha \in \mathcal{H} \otimes \mathcal{H}$$

$$\rightarrow \Phi(|\psi\rangle\langle\psi|) = \sum_\alpha tr_2 \left(1 \otimes |e\psi\rangle\langle e\psi|\right) |s_\alpha\rangle\langle s_\alpha|$$

Fix an α and write $|s\rangle$ for $|s_\alpha\rangle$: $\quad s = \sum_{m,n} z_{mn}\, e_m \otimes e_n$

$$tr_2 \left(1 \otimes |e\psi\rangle\langle e\psi|\right) |s\rangle\langle s|$$

$$= \sum_{mn\,pq} z_{mn} \bar{z}_{pq}\, tr_2 \left(1 \otimes |e\psi\rangle\langle e\psi|\right) |e_m \otimes e_n\rangle\langle e_p \otimes e_q|$$

$|e_m \otimes e_n\rangle\langle e_p \otimes e_q| = |e_m\rangle\langle e_p| \otimes |e_n\rangle\langle e_q|$

$tr_2(\dots) = |e_m\rangle\langle e_p|\, \langle e\psi, e_n\rangle\langle e_q, e\psi\rangle$

$\langle e\psi, e_n\rangle = \langle e_n, \psi\rangle, \quad \langle e_q, e\psi\rangle = \langle \psi, e_q\rangle$

$$= \left(\sum_{mn} z_{mn} |e_m\rangle\langle e_n|\psi\rangle\right)\left(\langle\psi| \sum_{pq} \bar{z}_{pq} |e_q\rangle\langle e_p|\right)$$

$$= |K\psi\rangle\langle K\psi| \quad \text{with} \quad K = \sum_{mn} z_{mn} |e_m\rangle\langle e_n|$$

We have shown that

$$\Phi(|\psi\rangle\langle\psi|) = \sum_\alpha K_\alpha |\psi\rangle\langle\psi| K_\alpha^*$$

Now extend to all operators X on $\mathcal{B}(\mathcal{H})$ by linearity:

$$X = \operatorname{Re} X + i \operatorname{Im} X \qquad \begin{cases} \operatorname{Re} X = \dfrac{X+X^*}{2} \\ \operatorname{Im} X = \dfrac{X-X^*}{2i} \end{cases}$$

Both $\operatorname{Re} X$ & $\operatorname{Im} X$ are hermitian, so diagonalizable.

E.g. $\operatorname{Re} X = \sum_j \lambda_j |\psi_j\rangle\langle\psi_j|$. As Φ is linear we have $\Phi(X) = \sum_\alpha K_\alpha X K_\alpha^* \quad \forall X \in \mathcal{B}(\mathcal{H})$.

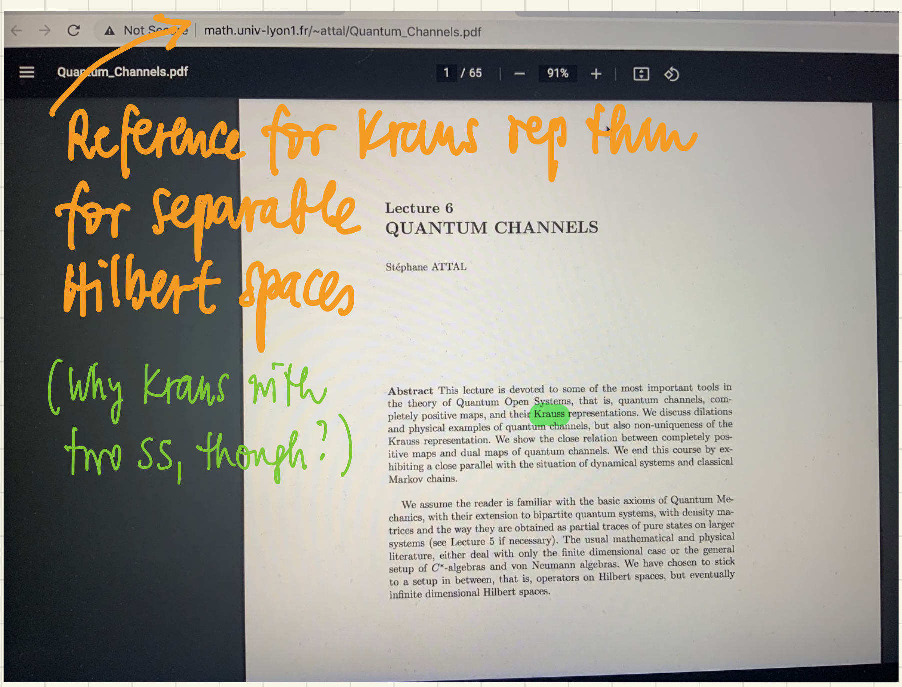
We have shown:

Theorem (Kraus representation theorem)

Suppose Φ is CPTP on $\mathcal{B}(\mathcal{H})$, $\dim \mathcal{H} = d < \infty$. Then there are operators $K_\alpha \in \mathcal{B}(\mathcal{H})$, $\alpha = 1, \dots, d^2$ s.t. $\forall X \in \mathcal{B}(\mathcal{H})$:

$$\Phi(X) = \sum_\alpha K_\alpha X K_\alpha^* \quad \text{and} \quad \sum_\alpha K_\alpha^* K_\alpha = \mathbb{1}$$

Conversely, if Φ is of this form for some operators K_α, then Φ is CPTP.



Lecture 6
QUANTUM CHANNELS

Stéphane ATTAL

Abstract This lecture is devoted to some of the most important tools in the theory of Quantum Open Systems, that is, quantum channels, completely positive maps, and their Krauss representations. We discuss dilations and physical examples of quantum channels, but also non-uniqueness of the Krauss representation. We show the close relation between completely positive maps and dual maps of quantum channels. We end this course by exhibiting a close parallel with the situation of dynamical systems and classical Markov chains.

We assume the reader is familiar with the basic axioms of Quantum Mechanics, with their extension to bipartite quantum systems, with density matrices and the way they are obtained as partial traces of pure states on larger systems (see Lecture 5 if necessary). The usual mathematical and physical literature, either deal with only the finite dimensional case or the general setup of C^*-algebras and von Neumann algebras. We have chosen to stick to a setup in between, that is, operators on Hilbert spaces, but eventually infinite dimensional Hilbert spaces.

<u>Recall</u>: We defined map Φ on $\mathcal{B}(\mathcal{H})$,

$$\Phi(X) = \operatorname{tr}_R \left[\mathcal{U} \, X \otimes |\Omega\rangle\langle\Omega| \, \mathcal{U}^* \right]$$

This has a physical meaning. ('Physical rep.')

Then we saw: Φ is CPTP and has a Kraus rep.

$$\Phi(\cdot) = \sum_\alpha K_\alpha (\cdot) K_\alpha^* \qquad \text{'Kraus form'}$$

Then we showed that any CPTP map on $\mathcal{B}(\mathcal{H})$ is of the Kraus form.

But does any CPTP map Φ have the 'physical' interpretation coming from the reduction of an extended complex (reservoir added)?

The answer is: YES!!

Theorem. Let Φ be a linear map on a Hilbert space $\mathcal{H}$, $\dim \mathcal{H} = d < \infty$. The following are equivalent:

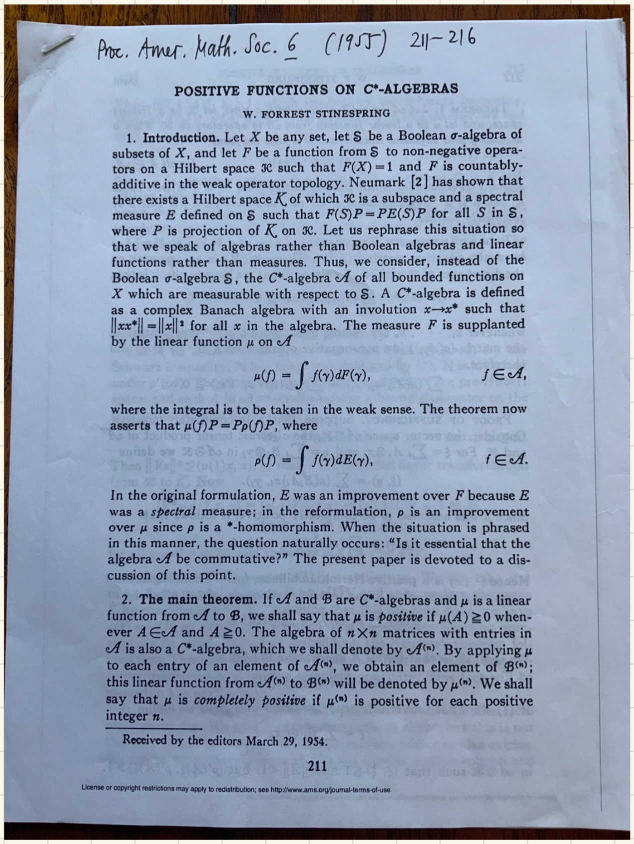
(1) Φ is CPTP

(2) There exists a Hilbert space $\mathcal{H}_R$, $\dim \mathcal{H}_R \leq d^2$, there exists a unit vector $\Omega \in \mathcal{H}_R$, the exists a unitary $\mathcal{U}$ on $\mathcal{H} \otimes \mathcal{H}_R$, s.t.

$$\Phi(X) = \mathrm{tr}_R\, \mathcal{U} \left(X \otimes |\Omega\rangle\langle\Omega| \right) \mathcal{U}^* \qquad \forall X \in \mathcal{B}(\mathcal{H})$$

There is an infinite-dimensional / C^* algebraic version:

Stinespring dilation theorem



Proc. Amer. Math. Soc. 6 (1955) 211–216

POSITIVE FUNCTIONS ON C*-ALGEBRAS

W. FORREST STINESPRING

1. **Introduction.** Let X be any set, let $\mathcal{S}$ be a Boolean σ-algebra of subsets of X, and let F be a function from $\mathcal{S}$ to non-negative operators on a Hilbert space $\mathcal{H}$ such that $F(X)=1$ and F is countably-additive in the weak operator topology. Neumark [2] has shown that there exists a Hilbert space $\mathcal{K}$ of which $\mathcal{H}$ is a subspace and a spectral measure E defined on $\mathcal{S}$ such that $F(S)P=PE(S)P$ for all S in $\mathcal{S}$, where P is projection of $\mathcal{K}$ on $\mathcal{H}$. Let us rephrase this situation so that we speak of algebras rather than Boolean algebras and linear functions rather than measures. Thus, we consider, instead of the Boolean σ-algebra $\mathcal{S}$, the C^*-algebra $\mathcal{A}$ of all bounded functions on X which are measurable with respect to $\mathcal{S}$. A C^*-algebra is defined as a complex Banach algebra with an involution $x \to x^*$ such that $\|xx^*\| = \|x\|^2$ for all x in the algebra. The measure F is supplanted by the linear function μ on $\mathcal{A}$

$$\mu(f) = \int f(\gamma)\, dF(\gamma), \qquad f \in \mathcal{A},$$

where the integral is to be taken in the weak sense. The theorem now asserts that $\mu(f)P = P\rho(f)P$, where

$$\rho(f) = \int f(\gamma)\, dE(\gamma), \qquad f \in \mathcal{A}.$$

In the original formulation, E was an improvement over F because E was a *spectral* measure; in the reformulation, ρ is an improvement over μ since ρ is a *-homomorphism. When the situation is phrased in this manner, the question naturally occurs: "Is it essential that the algebra $\mathcal{A}$ be commutative?" The present paper is devoted to a discussion of this point.

2. **The main theorem.** If $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras and μ is a linear function from $\mathcal{A}$ to $\mathcal{B}$, we shall say that μ is *positive* if $\mu(A) \geqq 0$ whenever $A \in \mathcal{A}$ and $A \geqq 0$. The algebra of $n \times n$ matrices with entries in $\mathcal{A}$ is also a C^*-algebra, which we shall denote by $\mathcal{A}^{(n)}$. By applying μ to each entry of an element of $\mathcal{A}^{(n)}$, we obtain an element of $\mathcal{B}^{(n)}$; this linear function from $\mathcal{A}^{(n)}$ to $\mathcal{B}^{(n)}$ will be denoted by $\mu^{(n)}$. We shall say that μ is *completely positive* if $\mu^{(n)}$ is positive for each positive integer n.

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We have already seen that if $\Phi(X) = \mathrm{tr}_R\, \mathcal{U}(X \otimes |\Omega\rangle\langle\Omega|)\mathcal{U}^*$ then Φ is CPTP. So (2) ⇒ (1) is proven.

Now we show that (1) ⇒ (2).

Task: given a CPTP map Φ on $\mathcal{B}(\mathcal{H})$ construct a Hilbert space $\mathcal{H}_R$, a unit vector $\Omega \in \mathcal{H}_R$ and a unitary $\mathcal{U}$ on $\mathcal{H} \otimes \mathcal{H}_R$ such that $\Phi(X) = \mathrm{tr}_R\, \mathcal{U}(X \otimes |\Omega\rangle\langle\Omega|)\mathcal{U}^*$.

Let Φ be given, CPTP on $\mathcal{B}(\mathcal{H})$, $\dim \mathcal{H} = d < \infty$. Then Φ has a Kraus rep.,
$$\Phi(X) = \sum_{\alpha=1}^{N} K_\alpha X K_\alpha^*, \quad N \leq d^2.$$

Set $\mathcal{H}_R = \mathbb{C}^N$ and let $\{e_\alpha\}_{\alpha=1}^N$ be an ONB of $\mathcal{H}_R$.

$\forall\, \psi, \chi \in \mathcal{H}$:

$$\begin{aligned}
\Phi(|\psi\rangle\langle\chi|) &= \sum_\alpha K_\alpha |\psi\rangle\langle\chi| K_\alpha^* \\
&= \sum_\alpha \mathrm{tr}_R\, K_\alpha |\psi\rangle\langle\chi| K_\alpha^* \otimes |e_\alpha\rangle\langle e_\alpha| \\
&= \sum_\alpha \mathrm{tr}_R\, |K_\alpha \psi \otimes e_\alpha\rangle\langle K_\alpha \chi \otimes e_\alpha|
\end{aligned}$$

We would like this to be of the form

$$\mathrm{tr}_R \; \mathcal{U} \; |\psi\otimes\Omega\rangle\langle\chi\otimes\Omega| \; \mathcal{U}^*.$$

Pick any $\Omega \in \mathcal{H}_R$, $\|\Omega\|=1$ and define linear map $\bar{\mathcal{U}}: \mathcal{H}\otimes\mathbb{C}\Omega \longrightarrow \mathcal{H}\otimes\mathcal{H}_R$ by

$$\bar{\mathcal{U}}\; \psi\otimes\Omega = \sum_\alpha K_\alpha\psi\otimes e_\alpha .$$

① $$\mathrm{tr}_R \; \bar{\mathcal{U}} \left(|\psi\rangle\langle\chi| \otimes |\Omega\rangle\langle\Omega| \right) \bar{\mathcal{U}}^*$$

$$= \mathrm{tr}_R \; |\bar{\mathcal{U}}\; \psi\otimes\Omega\rangle\langle\bar{\mathcal{U}}\;\chi\otimes\Omega|$$

$$= \sum_{\alpha,\beta} \mathrm{tr}_R \; |K_\alpha\psi\otimes e_\alpha\rangle\langle K_\beta\chi\otimes e_\beta|$$

$$= \sum_{\alpha,\beta} |K_\alpha\psi\rangle\langle K_\beta\chi| \; \mathrm{tr}_R \; |e_\alpha\rangle\langle e_\beta|$$

$$= \sum_\alpha K_\alpha |\psi\rangle\langle\chi| K_\alpha^*$$

$$= \Phi\left(|\psi\rangle\langle\chi|\right).$$

linearity

$$\Rightarrow \mathrm{tr}_R \; \bar{\mathcal{U}} \left(X\otimes|\Omega\rangle\langle\Omega| \right) \bar{\mathcal{U}}^* = \Phi(X), \quad \forall X\in\mathcal{B}(\mathcal{H}).$$

② $\langle \bar{\mathcal{U}}\, \psi \otimes \Omega,\ \bar{\mathcal{U}}\, \chi \otimes \Omega \rangle = \sum_{\alpha\beta} \langle K_\alpha \psi \otimes e_\alpha,\ K_\beta \chi \otimes e_\beta \rangle$

$$= \sum_{\alpha} \langle \psi,\ K_\alpha^* K_\alpha \chi \rangle$$

$$= \langle \psi, \chi \rangle$$

$$= \langle \psi \otimes \Omega,\ \chi \otimes \Omega \rangle.$$

So $\bar{\mathcal{U}} : \mathcal{H} \otimes \mathbb{C}\Omega \longrightarrow \mathcal{H} \otimes \mathcal{H}_R$ preserves inner products.

Exercise $\bar{\mathcal{U}}$ can be extended to a unitary $\mathcal{U}$ on $\mathcal{H}$.

This shows (1) ⇒ (2) of the theorem. ■

Part 3

Quantum dynamical semi-groups

Let $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_R$, $\dim \mathcal{H}_S = N < \infty$.

Let $H = H^* \in \mathcal{B}(\mathcal{H})$ and let ρ_R be a density matrix on $\mathcal{H}_R$.

Given any $t \in \mathbb{R}$, the map Φ_t defined by

$$X \mapsto \Phi_t(X) = \mathrm{tr}_R\left(e^{-itH}(X \otimes \rho_R)e^{itH}\right)$$

is a CPTP on $\mathcal{B}(\mathcal{H}_S)$. Moreover,

$$\Phi_{t=0} = \mathbb{1} = \mathrm{id}.$$

<u>Def.</u> A family $\{\Phi_t\}_{t\geq 0}$ of CPTP maps satisfying $\Phi_0 = \mathrm{id}$ is called a <u>CPTP dynamical map</u>.

S and R interact (via H) so generally $\Phi_{t+s} \neq \Phi_t \circ \Phi_s$.

<u>Def.</u> A CPTP family $\{\Phi_t\}_{t\geq 0}$ satisfying the group property $\Phi_{t+s} = \Phi_t \circ \Phi_s$, $s,t \geq 0$, and such that $t \mapsto \Phi_t$ is continuous, is called a <u>quantum dynamical semigroup</u> or, a <u>Markovian semigroup</u>.

Any (semi-) group has a generator. So

$$\Phi_t = e^{tL}, \qquad L = \frac{d}{dt}\Big|_{t=0} e^{tL}$$

where the generator L is an operator acting on $\mathcal{B}(\mathcal{H}_S)$.

Q : What is the general form of an L that generates a markovian semigroup?

As we show now, such L have a specific structure.

$\forall t$ fixed, Φ_t has a Kraus rep.

$$\Phi_t(X) = \sum_\alpha K_\alpha(t) X K_\alpha(t)^*.$$

Let $\{F_j\}_{j=1}^{N^2}$ be an ONB of $\mathcal{B}(\mathcal{H}_S)$,

$$\langle F_i, F_j \rangle \equiv \mathrm{tr}\, F_i^* F_j = \delta_{ij}$$

inner product of $\mathcal{B}(\mathcal{H}_S)$

Choose $F_{N^2} = \frac{1}{\sqrt{N}} \mathbb{1}$. Then all F_j for $j < N^2$

are traceless. We expand,

$$K_\alpha(t) = \sum_j \langle F_j, K_\alpha(t) \rangle F_j$$

$$\Rightarrow \quad \Phi_t(X) = \sum_{i,j=1}^{N^2} c_{ij}(t) \, F_i X F_j^*$$

$$\downarrow$$

$$c_{ij}(t) = \sum_\alpha \langle F_i, K_\alpha(t) \rangle \overline{\langle F_j, K_\alpha(t) \rangle}$$

$\forall X$:

$$LX = \lim_{t \to 0_+} \frac{1}{t} \left(\Phi_t(X) - X \right)$$

$$= \lim_{t \to 0_+} \left\{ \sum_{i,j=1}^{N^2-1} \frac{c_{ij}(t)}{t} F_i X F_j^* + \frac{1}{N} \frac{c_{N^2 N^2}(t) - 1}{t} X \right.$$

$$\left. + \frac{1}{\sqrt{N}} \sum_{i=1}^{N^2-1} \frac{c_{iN^2}(t)}{t} F_i X + \frac{1}{\sqrt{N}} \sum_{j=1}^{N^2-1} \frac{c_{N^2 j}(t)}{t} X F_j^* \right\}$$

Exercise: Show that each term in the sum converges individually as $t \to 0_+$.

So we define

$$a_{N^2N^2} = \lim_{t\to 0} \frac{c_{N^2N^2}(t) - N}{t}$$

$$a_{iN^2} = \lim_{t\to 0} \frac{c_{iN^2}(t)}{t}$$

$$a_{N^2j} = \lim_{t\to 0} \frac{c_{N^2j}(t)}{t} \qquad i,j = 1,\dots,N^2-1.$$

$$a_{ij} = \lim_{t\to 0} \frac{c_{ij}(t)}{t}$$

and also $F = \frac{1}{\sqrt{N}} \sum_{i=1}^{N^2-1} a_{iN^2} F_i$.

Then

$$LX = \sum_{i,j=1}^{N^2-1} a_{ij}\, F_i X F_j^* + \frac{1}{N} a_{N^2N^2} X + FX + XF^*.$$

With $\operatorname{Re} F = \frac{F+F^*}{2}$, $\operatorname{Im} F = \frac{F-iF^*}{2i}$ we get

$$LX = \sum_{i,j=1}^{N^2-1} a_{ij} \, F_i X F_j^* + \frac{1}{N} a_{N^2N^2} X + \{ \mathrm{Re}\, F, X \} + i [\mathrm{Im}\, F, X]$$

where $\{A, B\} = AB + BA$, $[A, B] = AB - BA$.

Φ_t is trace preserving $\Rightarrow$ $\mathrm{tr}\, LX = 0$

$$0 = \mathrm{tr} \left(\sum_{i,j=1}^{N^2-1} a_{ij} \, F_j^* F_i + \frac{1}{N} a_{N^2N^2} \mathbb{1} + 2\, \mathrm{Re}\, F \right) X = 0, \quad \forall X.$$

$$\Rightarrow \quad \mathrm{Re}\, F = -\frac{1}{2N} a_{N^2N^2} \mathbb{1} - \frac{1}{2} \sum_{i,j=1}^{N^2-1} a_{ij} F_j^* F_i.$$

Set $H = -\mathrm{Im}\, F = H^*$

$$\Rightarrow \quad LX = -i [H, X] + \sum_{i,j=1}^{N^2-1} a_{ij} \left(F_i X F_j^* - \frac{1}{2} \{ F_j^* F_i, X \} \right).$$

Exercise: Show that the coefficient matrix $A = (a_{ij})_{i,j=1}^{N^2-1}$ is positive.

We diagonalize A : $\exists$ unitary $\mathcal{U}$ and eigenvalues $\gamma_j \geq 0$,

$$A = \mathcal{U} D \mathcal{U}^*, \quad D = \mathrm{diag}\,(\gamma_1, \dots, \gamma_{N^2-1}).$$

Set $V_\ell = \sum_{i=1}^{N^2-1} \mathcal{U}_{i\ell} F_i$. Then the above formula for L gives the following result:

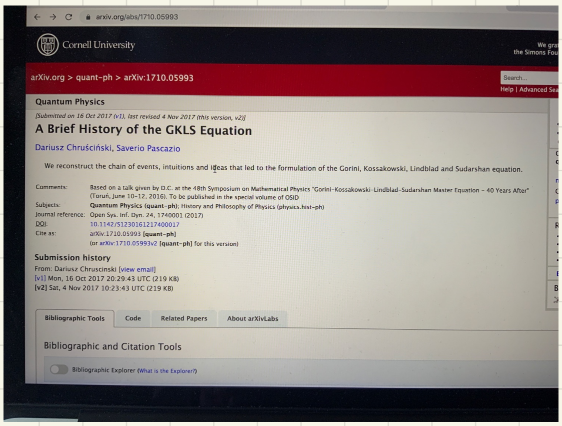
Theorem. If a linear operator L on $\mathcal{B}(\mathcal{B}(\mathbb{C}^N))$ is the generator of a CPTP semigroup, then there are operators $H = H^*$, $V_\ell \in \mathcal{B}(\mathbb{C}^N)$, $\ell = 1, \dots, N^2-1$ and $\gamma_\ell \geq 0$ s.t.

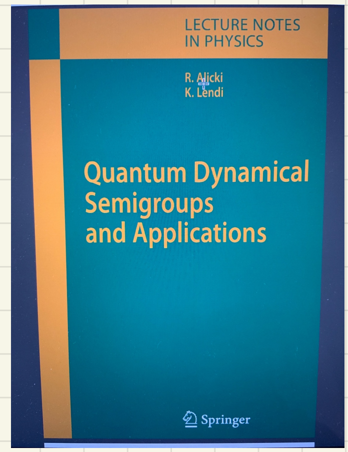
$$LX = -i[H, X] + \sum_{\ell=1}^{N-1} \gamma_\ell \left(V_\ell X V_\ell^* - \frac{1}{2} \{ V_\ell^* V_\ell, X \} \right).$$

The converse result holds as well. The theorem (and its converse) is called the Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) theorem.

Gorini-Kossakowski-Sudarshan: JMP <u>17</u>, 821 (1976) (dim $\mathcal{H}_S < \infty$)

Lindblad: CMP <u>48</u>, 119 (1976) (L bounded & dim $\mathcal{H}_S \leq \infty$)





Open System & Information Dynamics

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