SA QuTI South African Quantum Technology Initiative

NITheCS National Institute for Theoretical and Computational Sciences

INTERNATIONAL YEAR OF Quantum Science and Technology

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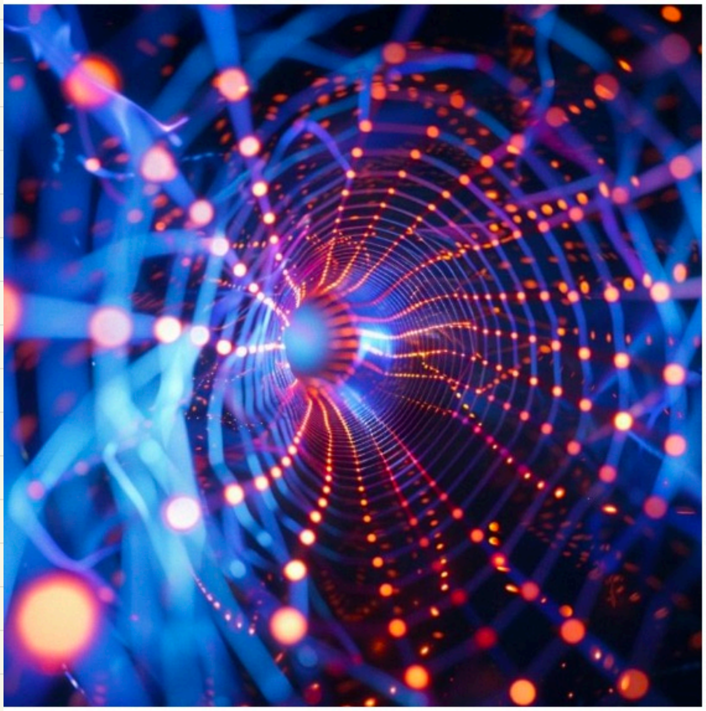

Theoretical Foundations of Quantum Science and Quantum Technologies

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Exercises for lectures on

Open quantum systems

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3. Suppose $\mathcal{H}$ is a Hilbert space, $\dim \mathcal{H} < \infty$.
Show that if $A \in \mathcal{B}(\mathcal{H})$ (bounded operators on $\mathcal{H}$) is such that $\operatorname{tr}(AX) = 0$ $\forall X \in \mathcal{B}(\mathcal{H})$, then $A = 0$.

4. Suppose M is a subspace of a Hilbert space $\mathcal{H}$ with $\dim \mathcal{H} = N < \infty$, and that $\mathcal{U} : M \to \mathcal{H}$ is a linear map s.t. $\langle \mathcal{U}x, \mathcal{U}y \rangle = \langle x, y \rangle$ $\forall x, y \in M$.

Show $\exists$ unitary $\bar{\mathcal{U}}$ on $\mathcal{H}$ s.t. $\bar{\mathcal{U}}\restriction_M = \mathcal{U}$.

<u>Sol</u>: Let $\{e_1, \dots, e_d, f_1, \dots, f_{N-d}\}$ be an ONB of $\mathcal{H}$ ($\{e_j\}_{j=1}^d$ ONB of M, $\{f_1, \dots, f_{N-d}\}$ ONB of $M^\perp$).

Then $\{\mathcal{U}e_j = \varepsilon_j\}_{j=1}^d$ are d orthonormal vectors in $\mathcal{H}$:

$$\langle \varepsilon_j, \varepsilon_k \rangle = \langle \mathcal{U}e_j, \mathcal{U}e_k \rangle = \langle e_j, e_k \rangle = \delta_{jk}.$$

Complete the set $\{\varepsilon_j\}_{j=1}^{d}$ to an ONB $\{\varepsilon_1, \dots, \varepsilon_d, \eta_1, \dots, \eta_{N-d}\}$ of $\mathcal{H}$.

Define $\bar{\mathcal{U}}$ by

$$\bar{\mathcal{U}} e_j = \varepsilon_j , \qquad j = 1, \dots, d$$

$$\bar{\mathcal{U}} f_k = \eta_k \qquad k = 1, \dots, N-d$$

This $\bar{\mathcal{U}}$ satisfies $\bar{\mathcal{U}}\restriction_M = \mathcal{U}$ (as the action of $\bar{\mathcal{U}}$ on the basis $\{e_j\}_{j=1}^{d}$ of M is the same as the action of $\mathcal{U}$) and $\bar{\mathcal{U}}$ is unitary: The columns $\{\varepsilon_1, \dots, \varepsilon_d, \eta_1, \dots, \eta_{N-d}\}$ of $\bar{\mathcal{U}}$ as a matrix are orthogonal.

(5) In the context of the quantum dynamical semigroups Φ_t, recall that the generator is given by

$$LX = \lim_{t\to 0_+} \left\{ \sum_{i,j=1}^{N^2-1} \frac{c_{ij}(t)}{t} F_i X F_j^* + \frac{1}{N} \frac{c_{N^2N^2}(t)-1}{t} X + \frac{1}{\sqrt{N}} \sum_{i=1}^{N^2-1} \frac{c_{iN^2}(t)}{t} F_i X + \frac{1}{\sqrt{N}} \sum_{j=1}^{N^2-1} \frac{c_{N^2j}(t)}{t} X F_j^* \right\}$$

For $i,j \in \{1, \dots, N\}$ define $\Gamma_{ij}(X) = F_i X F_j^*$.

(1) Show that the $\{\Gamma_{ij}\}_{i,j=1}^{N^2}$ are $N^2 \times N^2$ linearly independent elements of $\mathcal{B}(\mathcal{B}(\mathcal{H}))$.

Consequently, they are a basis of $\mathcal{B}(\mathcal{B}(\mathcal{H}))$.

Solution: Let U be the unitary that changes the ONB $\{F_\alpha\}_{\alpha=1}^{N^2}$ into the ONB $\{E_{ij}\}_{i,j=1}^{N^2}$, where E_{ij} is the matrix with entries zero except one 1 in the (i,j) spot. Then

$$\sum_{\alpha,\beta} c_{\alpha\beta}\, \Gamma_{\alpha\beta} X = 0 \iff \sum_{\alpha\beta} c_{\alpha\beta}\, F_\alpha X F_\beta^{*} = 0$$

$$\iff \sum_{\alpha\beta} c_{\alpha\beta}\, (UF_\alpha) X (UF_\beta)^{*} = 0$$

Now $\quad U F_\alpha = \sum_{ij} u_{(ij),\alpha}\, E_{ij}$

$$(UF_\beta)^{*} = \sum_{k,\ell} \overline{u_{(k\ell),\beta}}\, (E_{k\ell})^{*} = \sum_{k,\ell} \overline{u_{(k\ell),\beta}}\, E_{\ell k}$$

$$\iff \sum_{ij} \sum_{k\ell} d_{(ij),(k\ell)}\, E_{ij} X E_{\ell k} = 0$$

where $\quad d_{(ij),(k,\ell)} = \sum_{\alpha\beta} c_{\alpha\beta}\, u_{(ij),\alpha}\, \overline{u_{(k\ell),\beta}}$

Now $\quad (U^{*})_{\beta,(k\ell)} = \overline{(U)_{(k\ell),\beta}} = \overline{u_{(k,\ell),\beta}}$, so

$$d_{(ij),(k,\ell)} = (U C U^{*})_{(ij),(k,\ell)}, \quad C = (c_{\alpha\beta}).$$

Next, $E_{ij} X E_{\ell k} = |e_i\rangle\langle e_k| \quad X_{j\ell}$ (matrix element of X)

Choose $X = E_{rs}$, fixed r, s. Then

$$\sum_{\alpha,\beta} c_{\alpha\beta} \Gamma_{\alpha\beta} X = 0 \iff \sum_{ik} d_{(ir),(ks)} |e_i\rangle\langle e_k| = 0$$

$$\iff d_{(ir),(ks)} = 0 \quad \forall i, k$$

We can choose r, s arbitrary. So

$$\sum_{\alpha,\beta} c_{\alpha\beta} \Gamma_{\alpha\beta} \iff d_{(ij)(k\ell)} = 0 \quad \forall i j k \ell$$

$$\iff U C U^* = 0$$

$$\iff C = 0$$

$$\iff c_{\alpha\beta} = 0 \quad \forall \alpha, \beta.$$

This proves the $\Gamma_{\alpha\beta}$ are linearly independent.

In terms of the Γ_{ij}, we have (strong topology of $\mathcal{B}(\mathcal{B}(\mathcal{H}))$)

$$L = \lim_{t \to 0_+} \left\{ \sum_{i,j=1}^{N^2-1} \frac{c_{ij}(t)}{t} \Gamma_{ij} + \frac{1}{N} \frac{c_{N^2N^2}(t) - 1}{t} \Gamma_{N^2N^2} \right.$$
$$\left. + \frac{1}{\sqrt{N}} \sum_{i=1}^{N^2-1} \frac{c_{iN^2}(t)}{t} \Gamma_{iN^2} + \frac{1}{\sqrt{N}} \sum_{j=1}^{N^2-1} \frac{c_{N^2j}(t)}{t} \Gamma_{N^2j} \right\}.$$

For short, $L = \lim_{t \to 0_+} \sum_s \xi_s(t) \Gamma_s$ for some $\xi_s(t) \in \mathbb{C}$

and a basis $\{\Gamma_s\}$ of $\mathcal{B}(\mathcal{B}(\mathcal{H}))$. Then $L = \sum_s \lambda_s \Gamma_s$ for some (unique) $\{\lambda_s\} \subset \mathbb{C}$ and hence

$$\lim_{t \to 0_+} \sum_s \left(\xi_s(t) - \lambda_s \right) \Gamma_s = 0.$$

But as the Γ_s are linearly independent, $\exists c > 0$ s.t.

$$\left\| \sum_s \left(\xi_s(t) - \lambda_s \right) \Gamma_s \right\| \geq c \sum_s \left| \xi_s(t) - \lambda_s \right|$$

(basic linear algebra, see. e.g. [Kreyszig: Introductory Functional Analysis, Lemma 2.4-1])

Consequently, $\xi_s(t) \xrightarrow{t \to 0_+} \lambda_s \quad \forall s.$

6 Show that the coefficient matrix $A = (a_{ij})_{i,j=1}^{N^2-1}$ is positive, $A \geq 0$.